Embracing the Future or Building on the Past? Growth with New and Old Technologies

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Abstract

Is growth driven by the emergence of new paradigms or mostly through the perfection of existing technologies? And is the allocation of research effort between emerging technologies versus established ones efficient? To study these questions, I propose a new semi-endogenous growth model that incorporates technology vintages and the endogenous evolution of multiple technological paradigms through directed innovation. Despite the fact that technologies continuously emerge, making the state space unbounded, the model is remarkably tractable, allowing me to provide a comprehensive characterization of both the balanced growth equilibrium and the transitional dynamics. From a positive perspective, the model can rationalize two distinct empirical patterns of innovation over time and across technologies. Using two centuries of U.S. patent data, I first document that the age profile of patents has a pronounced hump-shape: the majority of contemporary patents are built upon technologies that are between 50 and 100 years old. Second, this age profile has remained remarkably stable throughout the past century. From a normative standpoint, the theory underscores a misallocation of research effort induced by the tendency among profit-maximizing firms to overinvest in further developing mature technologies. This fundamental inefficiency yields a suboptimally slow development of emerging technologies near the technological frontier. An estimated version of my model implies that transitioning from a laissez-faire equilibrium to the efficient allocation would increase the average growth rate of the economy from an annual 2% to 2.18% over the course of a century. These results shed new light on policy discussions concerning the prioritization of emerging technologies versus established ones. For instance, they provide a rationale for public policy to support investments in cutting-edge technologies, such as quantum computing or metabolic engineering.

Keywords: Emerging Technologies, R&D Misallocation, Growth.

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1 Introduction

Emerging technologies receive significant attention from society and policymakers. Consider, for example, the National Science Foundation (NSF), a prominent federal agency in the United States. For the fiscal year 2024, the NSF has requested a budget of over $11 billion, earmarking funds for several cutting-edge technologies, such as quantum computing and metabolic engineering.[1] This raises important questions: Should governments and public agencies provide *selective* support to the development of new technologies instead of older and mature ones? While mature technologies may lack the potential of their younger counterparts, they are widely adopted and play a significant role in the economy. Therefore, it is essential to understand whether and why market incentives are insufficient for the optimal allocation of research across these different fields.

In this paper, I propose a novel innovation-driven growth model with different technological vintages. Within this framework, I study how R&D investments are directed towards developing technologies of different ages. The allocation of R&D between new and old technologies predicted by my model matches both qualitatively and quantitatively a set of empirical patterns about technology and innovation within technology classes, which I document for the first time. Crucially, the theory allows me to examine the inherent misallocation of R&D across new and old technologies and to explore opportunities for policy intervention.

I begin by documenting how the US economy has balanced innovation between new and old technologies over time. Drawing on two centuries of patent data and using the US Patent Office's technological classification, I present two new facts. First, the cross-sectional distribution of innovation effort, as measured by the flow of new patents, across technologies of different ages has a pronounced hump shape. For example, in 2000, most patents built on technologies that emerged 70-100 years earlier, such as 'Wave transmission lines and networks'. Fewer patents built on newer technologies from the last decades of the 20th century, such as 'Software development', or on technologies from the 19th century or earlier. Second, this hump-shaped distribution is stationary and remained remarkably stable throughout the entire 20th century. For instance, in 1900, nascent technologies like 'Wave transmission lines and networks' commanded a patent share comparable to what 'Software Development' would achieve in 2000, a century later.

Next, I introduce a growth model that can rationalize these facts and be used to analyze whether the allocation of innovation efforts across technologies of varying maturity stages is efficient. In my theory, new and superior technologies emerge over time at the frontier. However, they face competition from long-established technologies that have evolved over time to become highly efficient and streamlined. The ensuing competition is not limited to sales and market share; it also involves a race for scarce innovation resources. Namely, profit-maximizing researchers who decide which technology to focus their research on.

The first key building block of the model, therefore, is the vintage structure: new and superior technologies emerge over time in the frontier. The arrival of a technology is represented by the

[1] See the NSF 2024 Fiscal Year Budget Request https://new.nsf.gov/about/budget/fy2024.

exogenous emergence of a new set of knowledge, for example, due to a breakthrough innovation that opens a new field. Embodying this knowledge, new products are endogenously created over time, becoming tangible representations of the technology. The greater the knowledge that a technology emerges with, the more productive the products embodying it will be. I refer to this knowledge as the technology's *inherent productivity*, representing its fundamental principles and characteristics.

To illustrate the concept of newly emerging vintage technologies that my notion seeks to capture, consider the example of magnetic tape recording technology. Fritz Pfleumer, a German engineer, invented the magnetic tape in 1928. He used the principle of magnetic encoding to store data as patterns of magnetic fields on the tape's surface. This approach was fundamentally different from earlier methods that used physical holes, notches, or mechanical positioning to encode data. Pfleumer's breakthrough was the creation of a set of knowledge — how to encode data using magnetic fields. Incorporating Pfleumer's key ideas, several magnetic tape models promptly emerged in the coming years, such as the Magnetophon, created by AEG Company in 1935 to store audio, or the BASF company 'LH' series of tapes.

I assume that each newly emerging technology has a higher inherent productivity than its predecessors. For example, magnetic tapes surpassed the efficiency of conventional punch card systems. Later, another revolution came in 1953 when IBM engineers introduced the concept of Random Access Storage with the first hard disk drive (HDD). Unlike the sequential data handling of magnetic tapes, this new paradigm of data storage allowed users to access data in any order, drastically boosting data retrieval efficiency.

The second key building block of the model is the cumulative nature of knowledge within a technology. Each new product that is created not only incorporates the knowledge from the technology to which it is linked but also contains its own original ideas. The latter, in turn, contribute to the expansion of the technology stock of knowledge. This enables future inventors to leverage a larger knowledge base in order to create superior products ("Standing on the shoulders of giants"). Over time, both the quantity and the average quality of varieties associated with a technology increase, signifying its process of refinement and underlying the gradual expansion in the expenditure share it commands.

Specifically, I assume that new products build and improve upon the average quality of all products associated with a particular technology. This formulation has two implications. First, it gives rise to a "process of perfection", whereby the average quality is increasing over time. Second, and more importantly, it implies that knowledge accumulation *within* a technology runs into diminishing returns. When a technology emerges, there are only a few products associated with it. Hence, new varieties have a large effect on average quality. Over time, as technologies mature, average quality is mostly determined by the stock of existing products, and the marginal quality increase due to new varieties is small. This means that to maintain a steady rate of improvement in both the number and the quality of varieties related to a given technology, an increasing research effort is needed over time, echoing the findings from Bloom et al. (2020) that ideas are getting harder to find.

The third key building block of the model is directed technical change across technology vintages. Profit-maximizing researchers face a trade-off between directing innovation towards new technologies, whose intrinsic potential is higher, or further pushing the development of older technologies that are already very productive and for which the Standing on shoulders of giants' effect is stronger. From this trade-off emerges a stationary distribution of R&D effort across technologies of different ages which is stationary in the economy Balanced Growth Path.

A challenge in the analysis is the ever-growing state space. Old technologies lose salience but never disappear in my theory. In spite of this, the model remains tractable: I analytically characterize the stationary distribution of R&D across technologies and examine the economic forces determining its shape. Moreover, I can also characterize the transitional dynamics, which is very important for the empirical analysis and the quantitative normative implications of the theory.

Two key forces play a pivotal role in shaping the distribution of research effort across technology vintages. The first is the (exogenous) rate at which new technologies inherently become more productive. The second is the endogenous rate at which the perfection process of technologies occurs, which depends on the extent of an intertemporal knowledge spillover within each technology vintage. On the one hand, since vintages emerging at the frontier exhibit higher productivity, the faster the growth potential of new relative to old technologies, the more the distribution mode of research effort shifts toward young vintages. On the other, within-technology knowledge spillovers yield an opposing effect. A stronger Standing on the shoulders of giants' force leads to a faster accumulation of knowledge stock in aging technologies, making them attractive to researchers despite the advent of superior technologies at the frontier. The resulting equilibrium distribution of research effort across vintages results from the balance between these two forces yielding a hump-shaped relationship between new innovations (patents) and the age of introduction of technologies that lines up with the data.

My model underscores a key inefficiency in research allocation under Laissez-faire conditions. While it is well established that the Standing on the shoulders of giants' effect results in innovation underinvestment — due to private agents not fully internalizing the benefits of their innovations on future researchers — I uncover a novel mechanism. In an economy where individual technologies eventually lose momentum and are gradually replaced by newer vintages, although the social value of innovation exceeds the private value within all technologies, this gap is relatively more pronounced in newer technologies. Within them, advances made by current innovation largely impact the knowledge base on which scientists build in the following years. This effect is smaller for an old technology because fewer scientists benefit from it as the rate of innovation declines. This asymmetry in knowledge spillovers implies that a social planner would allocate a higher share of total R&D resources to younger technologies in comparison to the market equilibrium allocation.

The model lends itself naturally to an empirical evaluation. First of all, I show that it can reproduce a number of qualitative patterns in the data. These include not only the hump-shaped cross-sectional distribution discussed above, but also the life cycles of innovation and sales traversed by individual technologies, from emergence to obsolescence. Notably, the model can mirror the

widely documented S-shaped adoption curve of technologies (Comin and Hobijn, 2004).

Motivated by these observations, I employ data moments to calibrate the parameters central to my theoretical framework. Three parameters stand out. The first pertains to the strength of knowledge spillovers, which steers the *within*-technology growth rate. I show that it can be identified with information on the stock market valuation of patents. In particular, the model predicts a high value for such a parameter if new patents related to a technology become more valuable as it ages. The second parameter governs the inherent rise in productivity for nascent technologies, and thus the *between*-technology growth rate. I show that its calibration is informed by observed aggregate growth rates. Lastly, congestion forces in research, capturing, for example, the dispersion in the comparative advantage of scientists across different fields, can be identified with joint variation on patent valuation and patenting activity within a technology.

While in my semi-endogenous growth model policy does not influence long-term growth, it can significantly impact transitional growth, as well as the long-term levels of GDP and welfare. This is particularly true given the model's prediction of very gradual and persistent transitions. To gauge the quantitative productivity and welfare losses from underinvestment in new technologies, I use the calibrated model to perform counterfactual exercises. I find that implementing the socially optimal research allocation initially leads to a slowdown in growth, as it channels more resources into newer, initially less productive technologies. However, the decline is temporary. After 15 years, growth under the optimal allocation exceeds growth under Laissez-faire. The gains are sizeable. Over a century, the average annual growth rate is 2.18% compared to 2% under Laissez-faire. Notably, the gains compound to a doubling of GDP in the long run. Welfare gains are also significant: with a discount rate of 2.5%, welfare increases in consumption equivalent units by 10%. These numbers are significant, especially when considering that only the allocation of research is changed in my counterfactuals, but total R&D effort is held fixed. These results provide a strong rationale for selectively supporting research on new technologies.

Related Literature Theories of growth and vintage technologies, or capital, have a long tradition in economics following the seminal works of Arrow (1962) and Solow (1960). Vintage models have been used in the context of technology diffusion, as was the case of Jovanovic and Lach (1989), Chari and Hopenhayn (1991), Atkeson and Kehoe (2007), Comin and Hobijn (2010), and Jovanovic and Yatsenko (2012). While this extensive literature has contributed to our understanding of the persistence in the adoption of old technologies and in the accumulation of physical and human capital associated with them, the possibility of endogenously directing innovation to older vintages, in the same way as to newer vintages, was not considered. In my model, both old and new technologies continue to improve through endogenous R&D, and the rise and decline of each of them are endogenously determined by this innovation race.

In this sense, I bridge the gap between the vintage technology literature based on Arrow (1962) and the modern innovation-driven theories of growth building on Romer (1990), Grossman and Helpman (1991), Aghion and Howitt (1992), and more recently Klette and Kortum (2004), and

Akcigit and Kerr (2018). Relative to the latter literature, when introducing the vintage structure, I show that research spillovers are asymmetric depending on the life cycle stage of a technology in which an innovation happens. This mechanism also differentiates my work from the Directed Technical Change Literature (Acemoglu, 2002; Acemoglu and Zilibotti, 2001; Acemoglu et al., 2016a). This literature studies the direction of innovation between two research lines and does not feature the endogenous cycle of ascendancy and obsolesce for different technologies over time.

In exploring the differentiation between innovation in emerging and mature technologies, this paper draws a parallel with Akcigit et al. (2020) framework, which considers basic and applied research within an endogenous growth model. They introduce this distinction through three key channels. Firstly, basic research enables spillovers that transcend the targeted industry, affecting multiple sectors, unlike applied research. Secondly, basic research may produce knowledge not immediately translatable into consumer products. Thirdly, they hypothesize that innovations originating from basic research enhance the efficiency of subsequent applied research within the same field. In my paper, however, the key distinction between alternative modes of innovation is solely the vintage structure of technology. I show that there is an intrinsic misallocation of R&D in a context where new and old technologies compete for innovation throughout their life cycles, even without assuming that the nature of spillovers differs when innovation occurs in old versus new technologies or that the degree of market appropriability (the conversion of research into profitable products) differs. Thus, my work contributes a distinct perspective to understanding the allocation of innovation resources in an evolving technological landscape.

My paper is also related to the empirical literature on technology diffusion (Griliches, 1957; Mansfield, 1961, 1963; David, 1990; Comin and Hobijn, 2010). My theory rationalizes in a growth model many of the empirical findings of this literature, such as the S-shaped diffusion pattern, according to which the path followed by the adoption shares of a technology has an S-shape. Moreover, while these empirical papers have focused on measures of consumption and production to assess the diffusion of technologies, I bring new evidence and theory on the *innovation* dynamics technologies experienced throughout their diffusion and declining paths.

My work is closely related to theories of General Purpose Technology (GPT) as conceptualized by Helpman and Trajtenberg (1994, 1996). In their framework, a technology serves as a platform for the creation of new varieties or applications over time. Similarly, in my theory, a technology is modeled akin to a GPT in Helpman and Trajtenberg (1994). However, they focus on the innovation happening only on the newest GPT, studying how it replaces an older GPT previously established. All innovation is allocated to the latest GPT, precluding an analysis of resource allocation across different technologies, which is a key aspect of this paper. Moreover, in Helpman and Trajtenberg (1994, 1996), there are no knowledge spillovers within a technology: current R&D does not impact the future cost of innovation on a technology. In contrast, I demonstrate that incorporating these spillovers is essential to replicate key empirical regularities, such as the hump-shaped distribution of innovation efforts across technologies and the S-shaped adoption curves.

Finally, my paper is related to a growing literature on R&D misallocation. For a long time,

the growth theory investigated the potential inefficiencies arising from the overall underinvestment in innovation. The key question was whether or not more real resources should be dedicated to innovation. This new literature on R&D misallocation focuses on how inefficiencies may exist even if we condition on a fixed amount of innovation resources (König et al., 2022; Acemoglu, 2023; Akcigit et al., 2020; Liu and Ma, 2023). It is important to discuss the relationship between my paper and Liu and Ma (2023). As in my work, they highlight the asymmetric nature of research spillovers. They show how this emerges from the network across sectors in the innovation space. Depending on the position a sector has in the network, an innovation within it may create spillovers of different importance. In my case, the asymmetry of spillovers comes from the life cycle of technologies, i.e., the stage of its life cycle a technology is in, and not the position on the network.

Road Map The structure of the paper is as follows. Section 2 documents the hump-shaped distribution of patents across technologies of different ages. Section 3 presents the theoretical framework. Sections 4 and 5 describe the data and how I take the model to the data. Section 6 contains the main results on the productivity and welfare costs of research misallocation, while Section 7 concludes.

2 The Hump-Shape of US Innovation in the 20th Century

The life cycle of a technology, from emergence to obsolescence, is reflected in the pattern of innovation efforts directed toward it. To illustrate this, I select US patents that contain keywords related to the steam engine technology in their text. Figure 1 shows the annual number of such patents from the 1830s, when US Patent Office (USPTO) records began, to the 1970s, when matched patents became negligible. During the 19th century, a period marked by the widespread adoption of steam engines (Crafts, 2004), there was a corresponding upward trend in the number of related patents. By the turn of the century, the advent of technologies such as the electric motor and the internal combustion engine led to its gradual obsolescence (Devine, 1983). Figure 1 captures this transition, showing a steady decline in the issuance of new steam-related patents as the 20th century progressed. However, albeit declining, such patent flows remained significant for a prolonged period, illustrating the steam engine's ongoing innovation even during obsolescence.

Are the patterns identified in specific technologies like the steam engine applicable on a broader scale? To address this question, I now turn to analyze nearly nine million US patents, classified into over 400 technological classes by the USPTO.[2] This enables me to study how innovation was balanced between emerging and established technologies over nearly two centuries in the US.

To identify the emergence date for each of these 400 technology classes, I adopt the methodology outlined by Griliches (1957). In this seminal paper, Griliches was primarily focused on technology diffusion - measuring the extent to which firms were adopting new technologies. He noted the

[2]These correspond to the utility classes in the Patent Classification System adopted by the USPTO during the period of analysis (USPC). They are structured under the principle that "a class generally delineates one technology from another" (USPTO, 2012).

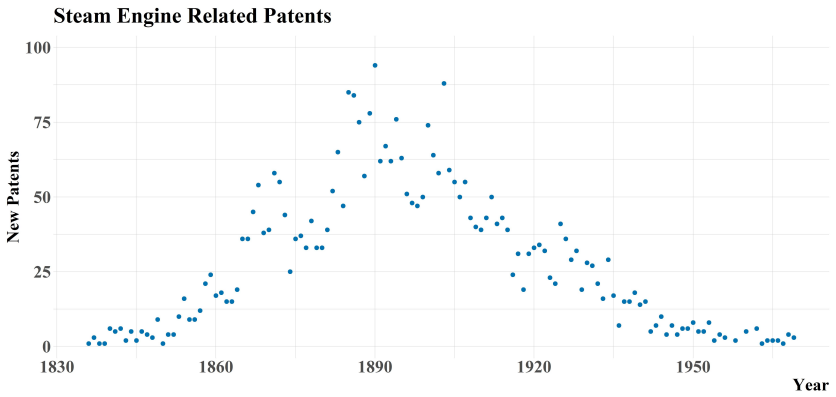

Figure 1: The Innovation Life Cycle - the Steam Engine case

Notes: The figure displays the annual count of patents identified as related to the Steam Engine technology, based on the presence of the term 'Steam Engine' (and the absence of terms associated with other motor types) in the text or title. The search criteria account for variations in keyword construction, such as plural forms and case sensitivity. I have narrowed the search to patents classified under the CPC class F04, 'Machines or Engines in general, Steam Engines,' as this classification comprehensively encompasses technologies related to steam engines.

characteristic S-shaped curve representing a technology's life cycle: an initial slow adoption phase, followed by a rapid uptake period, culminating in a plateau as the technology reaches peak adoption. Since S-shaped patterns align well with logistic functions, Griliches fitted a logistic trend tracing the adoption path of each technology. He defined the emergence date as the point at which this trend reached ten percent of its maximum value.

Here, I exploit the parallel between the life cycles of technology adoption and innovation, and employ Griliches' method to the patenting trajectory of each technology. This parallel is not only motivated by the case of the steam engine but is also corroborated by multiple case studies highlighting a consistent S-shaped pattern in patent timelines.[3] While a comprehensive exposition of this measurement procedure awaits in Section 4, here I present its main findings, which will shed light on the theoretical framework presented next in Section 3.

The left plot of Figure 2 displays the distribution of patents granted in the 2000s, categorized by the age of the associated technology classes. The horizontal axis denotes the age of these technologies, estimated using the Griliches (1957) methodology, while the vertical axis represents their corresponding share of the total patents issued in that decade. Each dot on the graph aggregates technology classes of the same estimated age, serving as a representation of a particular technology vintage. For illustration, the figure highlights three specific technology classes of different ages. For instance, the technology class 'Software Development', which emerged in the 1980s, falls into the category marked by the orange dot below its name, representing the 20-year-old technology vintage, along with other innovations that emerged in the same period. In contrast, 'Wave Transmission Lines and Networks' and 'Gas Heating and Illumination' represent older technologies, also

[3]See Achilladelis et al. (1990), Achilladelis (1993), Andersen (1999) and Haupt et al. (2007).

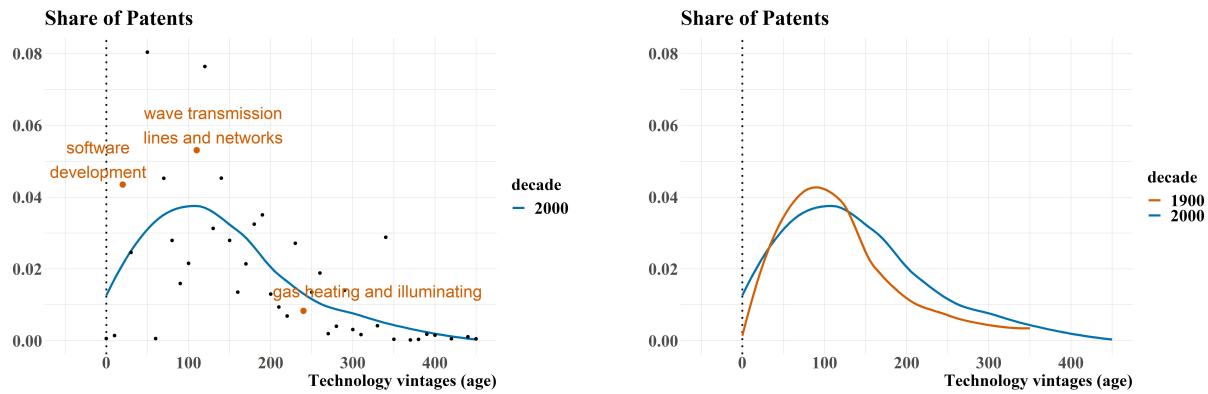

Figure 2: The Hump-Shape in Patent Distribution by Technology Age

Notes: The left plot shows the distribution of patents issued in the 2000s sorted according to the age of the technology class they are associated with. The age of each technology is estimated in accordance with Griliches (1957) methodology - and further detailed in Section 4. Each dot on the graph groups together technology classes of the same estimated age. The solid lines represent a non-linear fit, generated through local regression models from Cleveland et al. (1992).

highlighted in orange in the plot.

Importanlty, the left plot in Figure 2 shows the first key empirical result of this paper. There is a pronounced hump shape in the technology-age distribution of patenting at a point in time. In the 2000 decade, the majority of issued US patents built on technologies that emerged one century earlier, at the beginning of the 20th century, or late in the 19th century, such as 'Wave transmission lines and networks' and 'Drug, bio-affecting and body treating compositions'. There is also a significant volume of patents pertaining to older technologies that trace back much further in time. Even though older technology classes tend to accrue fewer patents as they age (evident from the declining right tail of the plot), this decrease is gradual. The overall volume of patents within this older technological space is comparable to that of patents associated with more recent technological advancements.

The right plot of Figure 2 contains a second key result: the hump-shape of patenting is remarkably stable throughout the 20th century. Like the left plot, this graph illustrates the technology-age distribution of patents in the 2000s, but also includes data from the 1900s decade for comparison. Not only is the hump shape evident in both periods, but strikingly the shape of the distribution is very similar: in 1900 as in 2000, most patents were issued in technological classes that were about 70-100 years old. For instance, in 1900, many technologies that emerged approximately one century before were tied to electricity, as 'Electrical generator or motor structure'. Technologies that would become the most important in 2000, were emerging around 1900, and had already some positive share in patenting by then, but were still a minority in terms of innovation shares.[4]

[4]One concern is that the stationary pattern observed in the right plot of Figure 2 may not be consistent if

The patterns in Figure 2 highlight that old technologies play a key role in aggregate innovation efforts. I will now present a theory that can rationalize these patterns. In Section 3, I will show that my model is quantitatively consistent with the observed hump-shape and highlight why the relative absence of innovation directed to new technologies indicates that the allocation of research in the decentralized equilibrium is inefficient.

3 Theory

This section presents a growth model with vintage technologies in the spirit of Arrow (1962) and the more recent theories of General Purpose Technology (Helpman and Trajtenberg, 1994, 1996). New technologies with increasing potential emerge over time. Within each technology, costly R&D effort creates new varieties, as in Romer (1990). Knowledge is cumulative within a technology: when innovating, researchers build on previous work to create higher-quality varieties. As such, the model also has a quality ladder element (Grossman and Helpman, 1991; Aghion and Howitt, 1992). Throughout time, therefore, the quantity and the average quality of varieties linked to a technology increase, representing its perfection process — and underlying the gradual expansion in the expenditure share commanded by it. In spite of the link to the Schumpeterian tradition, in my model, there is no creative destruction: new varieties with higher quality coexist with older ones within a technology class. Allowing new varieties to be more productive than old ones play a central role in my theory and in its empirical predictions.

3.1 The economy: environment and definitions

Consider an infinite-horizon economy in continuous time. The economy is populated by a representative household comprising a fixed measure of workers, denoted by L, and researchers, represented by R, who engage in R&D activities. The household derives utility from a unique consumption good C_t, discounts the future at rate ρ, and has preferences

$$U = \int_{t_0}^{\infty} e^{-\rho(t-t_0)} \ln C_t \, dt.$$

I denote by $\mathcal{T}_t$ the set of technologies available in this economy in time t. Each technology $\tau \in \mathcal{T}_t$ represents a unique pool of knowledge, which is embodied by an evolving set $\Omega(t|\tau)$ of intermediate varieties. Let $\mu(t|\tau)$ denote the measure $\mu(\Omega(t|\tau))$ of varieties associated with technology τ.

A competitive final good sector aggregates the intermediate varieties associated with existing

examined across other decades of the 20th century. In Section 4, I extend the comparative analysis beyond the 1900 and 2000 decades to include the distribution of patents in all other decades of the century. The established pattern demonstrates notable stability, remaining broadly consistent throughout the decades. Finally, another concern is that patents directed to old technologies may be of low quality, and overestimate their importance in comparison to newer technologies. In Section C in the Appendix, I replicate the entire analysis of Figures 1 and 2 using quality-weighted patents. For that, I use the indicator of patent importance developed by Kelly et al. (2021) and uniformly available for the entire sample period. The observed patterns are very similar. If anything, the stationary pattern becomes more apparent.

technologies and produces the final good Y_t as follows

$$Y_t = \frac{L^\beta}{1-\beta} \int_{\mathcal{T}_t} X(t|\tau) d\tau \tag{1}$$
$$\text{where } X(t|\tau) = \int_{\Omega(t|\tau)} z(\omega_\tau) x(\omega_\tau)^{1-\beta} d\omega_\tau.$$

Here, L corresponds to the aggregate labor input, inelastically supplied by the representative household. Meanwhile, $X(t|\tau)$ represents the intermediate input based on technology τ, delivered by the set of intermediate varieties that embody it. The quantity consumed of each of these varieties is denoted by $x_t(\omega_\tau)$, while $z(\omega_\tau)$ denotes its respective productivity level.

Each intermediate variety ω_τ has a variety-technology-specific productivity level. Formally

$$z(\omega_\tau) = A_\tau \times q(\omega_\tau),$$

where $q(\omega_\tau)$ represents ω_τ's own quality, and A_τ represents technology τ's intrinsic potential — incorporated by all varieties linked to it. For clarity, consider the Steam Engine technology. Individual varieties, such as the high-pressure engine used in steamboats or the stationary engine driving factory machines, each had their unique characteristics captured by $q(\omega_\tau)$. Nonetheless, they all incorporated the core fundamentals of the steam engine technology, such as the principle of converting steam pressure into mechanical motion.

Technology vintages Every period, one new technology appears exogenously. Therefore, the set of available technologies at calendar time t equals $\mathcal{T}_t = (-\infty, t]$. Hereafter, I will use the terms "technologies" and "vintages" synonymously, since $\tau \in (-\infty, t]$ also index the vintage of a technology by denoting the year in which it emerged.

The intrinsic potential of the technology that just emerged, A_t, stands for the knowledge frontier. As such, I assume it increases over time at rate $\gamma > 0$. Formally,

$$A_\tau = e^{\gamma\tau}, \tag{2}$$

for every technology τ. A high γ implies that each successive generation of technology has significantly higher intrinsic potential than its predecessors. In contrast, with a low γ, the differences between new and old technologies are less pronounced. As such, γ represents the incentives to "embrace the future" and plays a crucial role in the equilibrium of this economy.

Technology perfection After the emergence of a technology, follow-up innovations lead to its gradual perfection. Higher-quality varieties linked to it are created over time, representing the emergence of products embodying new knowledge to reduce costs, solve previous technical problems, improve design and form, or even represent new applications of the technology on which they build. Therefore, not only does the measure $\mu(t|\tau)$ of varieties associated with a technology τ increase, but

also their average quality, driven by the introduction of better varieties. To track this evolution, I define as $Q(t|\tau)$ the average quality of all available varieties at time t linked to a vintage τ. Formally,

$$Q(t|\tau) = \int q \, dF_{t,\tau}(q) \tag{3}$$

where $F_{t,\tau}(q)$ is the endogenous distribution of quality over the set of intermediate varieties $\Omega(t|\tau)$.[5]

I assume that if a variety is invented at time t within a technology $\tau \in (-\infty, t]$, its quality level q is influenced by (i) the quality of existing intermediates within such technology, and by (ii) an original component λ drawn from a distribution $H(\cdot)$ at the time of the invention.

Assumption 1. *The quality $q(\omega_\tau)$ of a variety ω_τ invented at t within a technology $\tau \in (-\infty, t]$ is*

$$q(\omega_\tau) = Q(t|\tau)\lambda,$$

where $\lambda \sim H(\cdot)$ and $\bar{\lambda} \equiv \int \lambda dH(\lambda) \geq 1$.

Assumption 1 states that innovation at time t builds on the knowledge stock already accumulated within the targeted technology, $Q(t|\tau)$, characterizing a standing on the shoulders of giants externality. If $\bar{\lambda} > 1$, the average quality of varieties related to technology τ will increase over time with the arrival of new and on average better products. This represents the *perfection* process of a technology: τ may have a high intrinsic potential A_τ, but its first varieties may have been of poor quality. Only with R&D effort over time can it be perfected and streamlined. The larger $\bar{\lambda}$, the larger the advances made by each generation of varieties in a given technology. As such, $\bar{\lambda}$ represents the incentives to "build on the past".

Importantly, despite the fact that new varieties are better on average, Assumption 1 implies that ideas become harder to find within a technology, echoing the empirical findings from Bloom et al. (2020): exponential growth in $Q(t|\tau)$ becomes harder to achieve over time. In other words, a given rate of growth for the average quality of a vintage τ requires an increasing number of researchers to be sustained. Although this point will be made formally later, it is worth discussing the intuition here. Suppose that a scientist, building on the shoulders of giants (as defined in Assumption 1), can produce one new variety per unit of time. This will impact the average quality significantly more when the pool of existing varieties is small (for example, when the technology has recently emerged) in comparison to when the set of available varieties is large and the elasticity of the average quality to the marginal invention is small.

Finally, I assume that the emergence of a new vintage τ is characterized by the exogenous introduction of a small 'number' $\mu(\tau|\tau) = \mu_0$ of pioneering varieties with an average quality of $Q(\tau|\tau) = Q_0$. This can be attributed to the result of serendipity. Once the economy is endowed with the basic knowledge and initial prototypes of this new technology $\{A_\tau, \mu(\tau|\tau), Q(\tau|\tau)\}$, endogenous innovation through R&D activity will gradually perfect, streamline, and exploit its potential, diffusing it in the variety-space.

[5] In Appendix B I provide a formal treatment for the distribution $F_{t,\tau}(q)$ and the measure $\mu(t|\tau)$.

Notation A few remarks concerning notation are important. It is possible to index the vintage of a technology not only by the year τ in which it was born, but also by its age a in a given period t, defined as $a = t - \tau$. The notation $x_t(a)$ should be understood as the variable x, at calendar time t, for the technology with age a at that time. Notice that, by definition, $\mu_t(a) = \mu(t|t-a)$ and $Q_t(a) = Q(t|t-a)$. I will use the change of variables between a and τ when convenient.

3.2 Equilibrium Production Allocation Given Technology

At a given time, optimal decisions by firms and households determine the allocation of consumption and production based on the available technologies and their associated set of varieties $[\Omega(t|\tau)]_\tau$.

The final good, Y_t, serves as the numeraire in this economy. Since its production occurs under perfect competition, it implies the usual isoelastic demand for intermediate varieties:

$$x_t(\omega_\tau) = L\left(\frac{p_t(\omega_\tau)}{z(\omega_\tau)}\right)^{-\frac{1}{\beta}}.$$

Each intermediate variety, in turn, is sold by a monopolist firm - who must use $\psi z(\omega_\tau)$ units of the final good to produce each unit of the input $x_t(\omega_\tau)$. The solution to the firm's profit maximization problem involves it charging $p(\omega_\tau) = z(\omega_\tau)$, assembling $x(\omega_\tau) = L$ units, and realizing profits:

$$\pi_t(\omega_\tau) = \beta z(\omega_\tau) L, \tag{4}$$

where the normalization $\psi = 1 - \beta$ has been imposed.[6]

I now characterize the equilibrium level of output. Combining the optimality conditions of firms, the final good sector production function (1), and the definition of $Q(t|\tau)$ in (3), we can express aggregate output in a transparent way, as shown in the Proposition below.

Proposition 1 (Aggregate output)**.** *In equilibrium, aggregate output is given by:*

$$\begin{aligned} Y_t &= \frac{L}{1-\beta}\int_{-\infty}^{t} e^{\gamma\tau} Q(t|\tau)\mu(t|\tau)d\tau \\ &= \frac{A_t L}{1-\beta}\int_{0}^{\infty} e^{-\gamma a} Q_t(a)\mu_t(a)da, \end{aligned} \tag{5}$$

where $A_t = e^{\gamma t}$ is the knowledge frontier.

*Proof.*See Appendix B. □

Proposition 1 expresses the level of output as a function of the existing technologies at time t and their current development stage. As such, it shows that the contribution of a specific vintage a (or $\tau = t - a$) to output depends on three factors: (i) how perfected it is, as shown by the average quality $Q_t(a)$ of products associated with it; (ii) how diffused it is, as shown by the measure of

[6]This only changes the following equations and results up to a constant written in terms of β and ψ - two parameters that are not central to the main mechanisms of the model.

products in the economy associated with it, $\mu_t(a)$; (iii) and how obsolete it is, $e^{-\gamma a}$: relative to the frontier, vintages face an obsolescence rate γ as they age.

Similarly, these same three factors determine the total expenditure on varieties associated with a given technology, which is defined as $S(t|\tau) \equiv \int_{\Omega(t|\tau)} p(\omega_\tau)x(\omega_\tau)\ d\omega_\tau$. From firm's optimality conditions and the definition of $Q(t|\tau)$ it can be shown that

$$S(t|\tau) \propto e^{\gamma\tau} Q(t|\tau)\mu(t|\tau). \tag{6}$$

Therefore, the higher the perfection level $Q(t|\tau)$, the measure of varieties $\mu(t|\tau)$, and the intrinsic potential of a technology, the higher the expenditure on a technology. An implication is that technologies that experience relatively fast growth in $Q(t|\tau)$ and in $\mu(t|\tau)$ gain market share over time, while those that exhibit relatively low growth will shrink.

Finally, the resource constraint imposes that total output net of expenditures on the production of varieties corresponds to consumption:

$$C_t = Y_t - M_t = \alpha Y_t, \tag{7}$$

where M_t is the expenditure on varieties[7] and $\alpha = \beta(2-\beta)$.

3.3 Innovation and Equilibrium Dynamics

In the previous section, I discussed the equilibrium allocation given the state of technology. In this section, I characterize the equilibrium evolution over time of technology that determines economic growth. In my theory, there are three key drivers of productivity growth. The first is the exogenous arrival of new technology vintages τ. The second is the 'horizontal' expansion of varieties within each technology τ, which is captured by the term $\mu(t|\tau)$. The third is the 'vertical' improvement in the quality of each technology τ, which is captured by the term $Q(t|\tau)$.

I assume that innovation is carried out by a fixed measure R of researchers. Researchers are distinct people from workers endowed with specific skills. They run 'projects' directed at improving specific technologies. These projects are intrinsically risky but the risk is pooled by risk-neutral financial institutions that allow researchers to earn nonstochastic returns. This is a common approach in the literature (see, for example, Acemoglu et al. (2012), Acemoglu et al. (2016a), and Liu and Ma (2023)). It simplifies the analysis insofar as it fixes the total amount of productive resources devoted to innovation allowing me to focus sharply on its allocation across technology vintages.

The output of research is ideas that are patented, and the patents are sold in competitive auctions to firms producing individual varieties. Thus, the value of each patent equals the present

[7] From the solution to the monopolist firms' problem, the total quantity of final goods demanded by them to produce can be computed

$$M_t = (1-\beta)A_t L \int e^{-\gamma a} Q_t(a)\mu_t(a)da.$$

discounted value of the perpetual profit stream it generates. Using (4) it can be written as:

$$v_t(\omega_\tau) = \int_t^\infty e^{\left(-\int_t^s r(v)\, dv\right)}\, \pi(\omega_\tau)\, ds = \beta z(\omega_\tau) L \times D_t, \tag{8}$$

where $D_t \equiv \int_t^\infty \exp\left(-\int_t^s r(v)dv\right) ds$. If the interest rate r is constant, then $D_t = r^{-1}$.

Innovation is directed: researchers choose which technology to target in their projects in each period t. Let $\bar{v}(t|\tau)$ denote the ex-ante expected value of a successful innovation within technology τ. It is the crucial object to determine research incentives towards different technologies at t. Combining (8) and Assumption 1, $\bar{v}(t|\tau)$ is given by:

$$\bar{v}(t|\tau) = \beta e^{\gamma\tau} Q(t|\tau) \bar{\lambda} L D_t. \tag{9}$$

Let $R(t|\tau)$ denote the density of researchers at time t targeting technology τ. The rate at which a successful idea is discovered by an individual researcher working on technology τ is assumed to be:

$$\eta R(t|\tau)^{-\epsilon}, \tag{10}$$

where ϵ represents a congestion force or a duplication externality as in Jones (1995). Due to the overlap of ideas, the likelihood that a researcher generates the next innovation diminishes as more individuals focus on the same technological field. Notice that the total flow of ideas related to technology τ is represented by $\eta R(t|\tau)^{1-\epsilon}$. At the aggregate level, provided the congestion force is moderate (i.e., $\epsilon < 1$), an increase in the number of researchers devoted to a technology correlates with a higher overall rate of idea generation, albeit with diminishing returns.

In equilibrium, researchers' allocation across various technologies must meet a non-arbitrage condition, ensuring that no individual researcher would benefit from shifting their focus to a different technology. The congestion force integrated into the model through Equation (10) effectively prevents corner solutions. Specifically, it precludes situations where all research is concentrated on a single technology or completely absent from another. This is because the absence of research in any given technology would lead to an infinite arrival rate of ideas, making it highly attractive for researchers. In this model, therefore, the adjustment of R&D happens at the intensive (but not at the extensive) margin. It means that as a technology becomes relatively obsolete and less profitable compared to others, the share of researchers working on it shrinks, and asymptotically converges to zero, but it remains at least infinitesimally positive at any given point in time t.[8]

[8] It's also important to note that alternative assumptions could yield similar results to this congestion effect. One such scenario involves researchers with heterogeneous abilities across different technological fields, embodying comparative advantage forces. After independently drawing research productive levels for different technologies from an extreme value multivariate distribution whose shape parameter is $1/\epsilon$, they sort into research fields considering both the value of a patent within each of them and their own abilities. In such case, ϵ would control the dispersion of comparative advantages of researchers' abilities towards different fields.

For every pair of technologies τ and τ', the allocation of research must satisfy:

$$\frac{R(t|\tau)}{R(t|\tau')} = \left(\frac{\bar{v}(t|\tau)}{\bar{v}(t|\tau')}\right)^{\frac{1}{\epsilon}}. \tag{11}$$

Equation (11) makes it clear how the congestion force ϵ represents, in equilibrium, the supply elasticity of research. When a technology becomes more profitable relative to others, it attracts a proportionally greater number of researchers, a response moderated by the value of ϵ. As ϵ approaches infinity, suggesting an extremely large congestion force, the distribution of research across technologies tends toward uniformity, with $R(t|\tau)/R(t|\tau')$ converging to 1. This implies that despite differences in profitability, various technologies receive a similar proportion of research. In contrast, when ϵ is close to zero, indicating minimal congestion force, research becomes highly concentrated in the most profitable technology, causing the ratio $R(t|\tau)/R(t|\tau')$ to diverge in the direction of the more valuable vintage.

Research allocation solution The optimal behavior of researchers and their resulting allocation across technologies is intrinsically forward-looking, as evidenced by $R(t|\tau)$'s dependence on $\bar{v}(t|\tau)$ in Equation (11). Remarkably, however, the density of researchers $R(t|\tau)$ working on a vintage τ can be solved as a function only of predetermined state variables. This highlights the model's robust *tractability* and sets the stage for solving the stationary equilibrium in a closed form. To understand this, observe that the value function $\bar{v}(t|\tau)$, detailed in Equation (9), isolates the forward-looking component in the term D_t, which is uniform across all technologies and thus does not affect the relative allocation of research across vintages. Combining Equations (9) and (11), we arrive at the following formulation:

$$R(t|\tau) = \left(\frac{e^{-\gamma(t-\tau)}Q(t|\tau)}{\bar{Q}_t}\right)^{\frac{1}{\epsilon}} R, \tag{12}$$

or, equivalently:

$$R_t(a) = \left(\frac{e^{-\gamma a}Q_t(a)}{\bar{Q}_t}\right)^{\frac{1}{\epsilon}} R, \tag{13}$$

where $\bar{Q}_t \equiv \left[\int_0^\infty \left(e^{-\gamma\tilde{a}}Q_t(\tilde{a})\right)^{\frac{1}{\epsilon}} d\tilde{a}\right]^{\epsilon}$ is an age-discounted aggregate quality index for the economy.

Comparing technologies of ages a' and a'', we can write the following relation linking R&D efforts in the two vintages:

$$\epsilon\left(\log R_t(a') - \log R_t(a'')\right) = -\gamma(a' - a'') + \log Q_t(a') - \log Q_t(a'') \tag{14}$$

Equations (13) and (14) underscore the interplay of two crucial factors influencing research allocation across various technology vintages: *obsolescence* and *accumulated knowledge (quality) stock*. The age of a technology significantly impacts its positioning relative to the frontier; older technologies, being further from the frontier, inherently possess lower efficiency compared to their younger counterparts. This dynamic naturally inclines research investment toward younger vintages, a ten-

dency intensified by the rate γ, which dictates the speed of frontier expansion. Conversely, older technologies boast a more substantial accumulated quality stock, having benefited from extended periods of development and refinement. This accumulated knowledge base provides a robust platform for researchers, and incentivizes innovation in higher-age technologies despite the relative underdevelopment of younger technology fields, which poses greater challenges in developing high-quality, well-polished new varieties.

Laws of motion Given the innovation possibilities frontier of the economy, it is possible to write the laws of motion of $\mu(t|\tau)$ and $Q(t|\tau)$ for each technology, for a given allocation of research $[R(t|\tau)]_\tau$. As shown in detail in Appendix B.1, a technology τ evolves in time following the differential equations:

$$\frac{\dot{\mu}(t|\tau)}{\mu(t|\tau)} = \eta \frac{R(t|\tau)^{1-\epsilon}}{\mu(t|\tau)} \tag{15}$$

$$\frac{\dot{Q}(t|\tau)}{Q(t|\tau)} = \eta(\bar{\lambda} - 1)\frac{R(t|\tau)^{1-\epsilon}}{\mu(t|\tau)}, \tag{16}$$

for all $t \geq \tau$ and with $\mu(\tau|\tau) = \mu_0$ and $Q(\tau|\tau) = Q_0$ given. Equations (15)-(16) reveal that the growth rates in the average quality and the number of varieties associated with a technology τ are proportional to its innovation rate per variety, represented as $R(t|\tau)^{1-\epsilon}/\mu(t|\tau)$. Crucially this implies that the productivity of research decreases as $\mu(t|\tau)$ increases. To sustain a constant growth rate for both $Q(t|\tau)$ and $\mu(t|\tau)$, the research effort focused on technology τ, $R(t|\tau)$, must increase over time. Using the terminology from Bloom et al. (2020), ideas get harder to find within technology τ.

Expressing the laws of motion for Q and μ as functions of age is beneficial because these age-specific schedules will remain constant along a Balanced Growth equilibrium, to be defined below. It can be shown that (see Appendix B.1):

$$\frac{\partial \mu_t(a)}{\partial t} = -\frac{\partial \mu_t(a)}{\partial a} + \eta R_t(a)^{1-\epsilon} \tag{17}$$

$$\frac{\partial Q_t(a)}{\partial t} = -\frac{\partial Q_t(a)}{\partial a} + Q_t(a)(\bar{\lambda} - 1)\eta \frac{R_t(a)^{1-\epsilon}}{\mu_t(a)}. \tag{18}$$

Finally, the dynamic problem of the household implies the familiar Euler equation and transversality condition:

$$\frac{\dot{C}_t}{C_t} = r_t - \rho, \text{ and } \lim_{t\to\infty}\left[\exp\left(-\int_{t0}^{t} r(s)ds\right) V_t\right] = 0, \tag{19}$$

where V_t is the value of corporate assets in this economy, representing the household's total wealth:

$$V_t = \int_\tau \int_{\omega_\tau} v_t(\omega_\tau)\, d\omega_\tau d\tau$$

3.3.1. Equilibrium Definition

A dynamic equilibrium can now be characterized. In this economy, equilibrium objects are cross-sectional distributions that evolve over time. In particular, the state variables in a given period t are the distributions of quality and varieties $[Q_t(a), \mu_t(a)]_{a=0}^{\infty}$, representing the technology landscape agents face at that point. They evolve over time as a result of the profit-maximizing distribution of research efforts $[R_t(a)]_{a=0}^{\infty}$ and the innovation possibilities frontier, summarized by the laws of motion (18)-(17). Formally:

Definition 1. (Equilibrium) A trajectory for the cross-sectional distributions of varieties, average quality, value functions and research, $[\mathbf{\Theta}_t]_{t\geq t_0}$ (where $\mathbf{\Theta}_t \equiv [Q_t(a), \mu_t(a), \bar{v}_t(a), R_t(a)]_{a=0}^{\infty}$), for aggregate variables $[Y_t, C_t, X_t]_{t\geq t_0}$, and for prices $[r_t, w_t]_{t\geq t_0}$ is an **equilibrium trajectory** if, given an initial condition for $[Q_{t0}(a), \mu_{t0}(a)]$, the following conditions are satisfied:

1. Given prices, all firms in the economy maximize profits: the value functions satisfy (8) and hence (9).

2. Given the value functions, researchers maximize expected returns: their allocation satisfies (13).

3. Given prices, the representative household maximizes utility: the Euler equation and the Transversality condition (19) are satisfied.

4. The resource constraints are respected and all markets clear: $\int R_t(a)da = R$, $C_t = Y_t - M_t$, and $L_t = L$.

5. Given the optimal allocation of researchers and an initial condition $[Q_{t_0}(a), \mu_{t_0}(a)]_{a=0}^{\infty}$, the trajectories for $Q_t(a)$ and $\mu_t(a)$ satisfy the laws of motion (18)-(17).

An equilibrium trajectory can be easily computed numerically given its recursive nature. To see that, notice that the aggregate supply of innovation resources is constant and the relative supply of research across vintages at time t depends only on $[Q_t(a)]_{a\geq 0}$, as shown in (13). Therefore, after combining (13) and the laws of motion (17)-(18), one is left with differential equations for $Q_t(a)$ and $\mu_t(a)$ that depend only on the current level of these distributions. Given an initial condition $[Q_{t_0}(a), \mu_{t_0}(a)]_{a=0}^{\infty}$, they can be solved forward in time, yielding a trajectory for the state variables $\{[Q_t(a), \mu_t(a)]_{a=0}^{\infty}\}_{t\geq t_0}$. This in turn pins down a trajectory for output and consumption, which are a direct function of the state at each period (see equations (5) and (7)). The growth rate of consumption pins down the interest rate and the value functions, which, in turn, determine the allocation of researchers consistent with the trajectory for the state variables.

Now I turn to define a stationary (or balanced growth) equilibrium. It is characterized by cross-sectional technology-age distributions of varieties and quality that are time-invariant. As a result, although each technology when looked individually traverses a life cycle of rise and obsolescence, this cycle is stationary: for instance, the currently 50-year-old technology will always appear the same in any time period, differing only in its intrinsic productivity.

Definition 2. (Stationary Equilibrium) A stationary equilibrium is an equilibrium trajectory that satisfies, for every $t \geq t_0$, $\partial_t Q_t(a) = \partial_t \mu_t(a) = 0$.

3.3.2. Stationary (or Balanced Growth) Equilibrium Characterization

The transition dynamics, initiating from any given technology level $[Q_{t_0}(a), \mu_{t_0}(a)]_{a=0}^{\infty}$, is not only readily computable numerically, but also the stationary equilibrium is amenable to analytical solution. This is the content of Proposition 2 that follows.

Proposition 2. *There exists a unique stationary (or balanced growth) equilibrium. It is such that:*

1.Aggregate variables grow at the rate γ:

$$\frac{\dot{Y}_t}{Y_t} = \frac{\dot{C}_t}{C_t} = \gamma \tag{20}$$

2.If $\bar{\lambda} > 1$, the stationary technology-age distribution of quality at any given t is:

$$Q(a) = \left\{ c_1 - c_2 e^{-\frac{1-\epsilon}{\epsilon}\gamma a} \right\}^{\frac{1}{\frac{1}{\bar{\lambda}-1} - \frac{1-\epsilon}{\epsilon}}} \tag{21}$$

where c_1, c_2 are uniquely determined constants given by:

$$c_2 \equiv \frac{Q_0^{\frac{1}{\bar{\lambda}-1}} \eta(\bar{\lambda}-1)}{\mu_0 \frac{1-\epsilon}{\epsilon} \gamma \bar{Q}^{\frac{1-\epsilon}{\epsilon}}} \left(\frac{1}{\bar{\lambda}-1} - \frac{1-\epsilon}{\epsilon} \right), \quad c_1 \equiv Q_0^{\frac{1}{\bar{\lambda}-1} - \frac{1-\epsilon}{\epsilon}} + c_2,$$

and $\bar{Q}$ is the stationary level of $\bar{Q}_t$, which is unique. If $\bar{\lambda} = 1$, then $Q(a) = Q_0 = \lim_{\bar{\lambda}\to 1} Q(a)$.

3.If $\bar{\lambda} > 1$, the stationary technology-age distribution of varieties at any given t is:

$$\frac{\mu(a)}{\mu_0} = \left(\frac{Q(a)}{Q_0} \right)^{\frac{1}{\bar{\lambda}-1}}. \tag{22}$$

Conversely, if $\bar{\lambda} = 1$, then $\mu(a) = c_3 - c_4 e^{-\frac{1-\epsilon}{\epsilon}\gamma a} = \lim_{\bar{\lambda}\to 1} \mu(a)$, where:

$$c_4 \equiv \eta \left(\frac{\gamma}{\epsilon} \right)^{1-\epsilon} \frac{1}{\gamma^{\frac{1-\epsilon}{\epsilon}}}, \text{ and } c_3 \equiv \mu_0 + c_4.$$

*Proof.*See Appendix A.1. □

Proposition 2 contains two important implications. The first is that, in the stationary equilibrium, the arrival of new technologies is the sole driver of (long-run) growth. Since they arrive exogenously, this is a model of semi-endogenous growth (Jones, 1995). In the long run, the economy grows at the rate γ, which is the rate at which new technologies are inherently superior to older ones. The process by which technologies are perfected over time contributes to progress, but is not

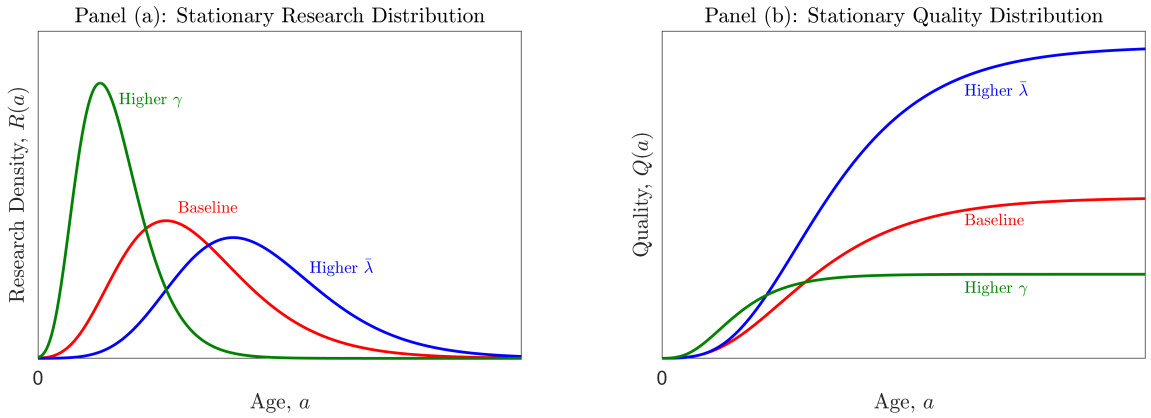

Figure 3: Cross-sectional technology-age stationary distributions for research and quality

Notes: The figure displays the stationary distributions of research $R(a)$ (Panel a) and average quality $Q(a)$ (Panel b) across technologies of different ages. Each plot shows the baseline calibration in red, the case of a higher frontier growth rate γ in green and the case of a higher step size on quality improvements $\bar{\lambda}$ in blue.

a source of growth in the long run. The reason is that there is a ceiling to the productivity of each technology. This stems from the fact that the productivity of research within each vintage (in the quality ladder sense) declines over time. As a result, the development of an existing technology loses momentum: as new technologies with a higher productivity keep arriving, the profitability of research perfecting old vintages is progressively dominated by doing research on newer vintages. The share of researchers directing their efforts to the older technology eventually declines to zero, i.e., $\lim_{a\to\infty} R(a) = 0$. Thus, in the long run, the productivity of each technology converges to an upper bound where $\lim_{a\to\infty} Q(a) = \bar{Q} < \infty$, and $\lim_{a\to\infty} \mu(a) = \bar{\mu} < \infty$.

Figure 3 illustrates this dynamics. Exploiting Proposition 2's analytical characterization, it presents the stationary distributions of research (Panel a) and average quality (Panel b) across technologies of different ages. In each plot, I show a baseline calibration in red, and two alternative calibrations in which I either increase the frontier growth rate γ (green curve), or the knowledge spillover parameter $\bar{\lambda}$ (blue curve). Notice in Panel (a) how, for all calibrations, the share of research $R(a)$ shrinks to zero as the age of technologies increases. In parallel, Panel (b) shows that, for such old technologies, the average quality stops growing, indicating the end of the perfection process of very old technologies.

Although the share of research in old technologies decreases to zero, whether it first increases when technologies are young hinges on the model parameters. This is the second important implication of Proposition 2, and is discussed with the help of Corollary 2.1 below.

Corollary 2.1 (Innovation in the frontier)**.** For any vector of parameters, there exists $\underline{b} > 0$ (where $\underline{b}$ is a function of the parameter vector) such that, if $\left(\bar{\lambda} - 1\right)/\gamma < \bar{b}$, then $R'(0) < 0$, whereas, if $\left(\bar{\lambda} - 1\right)/\gamma \geq \bar{b}$, then $R'(0) \geq 0$.

*Proof.*See Appendix A.1. □

Corollary 2.1 shows that, conditional on the other parameters, a sufficiently high ratio $\left(\bar{\lambda}-1\right)/\gamma$ implies that initially the share of researchers targeting a technology increases. This results from $\bar{\lambda}-1$ being relatively higher than γ. The higher $\bar{\lambda}-1$, the higher the productivity growth within each vintage (in the quality ladder sense). In turn, a low γ indicates that the productivity of vintages emerging in the frontier is growing slowly. Therefore, the ratio between the speed at which productivity growth happens within-vs-across technologies is key. If the former is sufficiently higher than the later, i.e., $\left(\bar{\lambda}-1\right)/\gamma$ is large, a technology, after emerging, experiences faster productivity growth than the frontier, and the productivity (and profitability) gap separating it from the latest vintage increases. As a result, the share of research directed to the young technology increases as it ages. On the other hand, if $\left(\bar{\lambda}-1\right)/\gamma$ is small, the productivity growth within a technology starts at a low level and cannot keep pace with the frontier growth rate. It loses research share from the beginning.

In light of the discussion of Corollary 2.1, the calibration in all plots of Figure 3 is such that $\left(\bar{\lambda}-1\right)/\gamma$ is sufficiently high, implying that research shares initially increase with age before declining as technologies get old. Turning attention to Panel (a) and its baseline calibration (depicted by the red line), we observe the research distribution $R(a)$. Notably, the peak of this distribution aligns with middle-aged technologies. In contrast, the research directed towards nascent technologies (close to age 0) is notably smaller, as is the share directed to technologies of advanced ages. Introducing a higher balanced growth rate γ (illustrated by the green curve) brings about a shift. While the pattern retains its qualitative essence, the peak of the distribution skews left, highlighting that more researchers are drawn to emerging technologies, as they are now relatively more productive than in the baseline scenario. In contrast, the blue curve shows that the opposite shift happens if we consider an increase in the knowledge spillover parameter $\bar{\lambda}$: the mode of the distribution shifts significantly to the right. Once again, the explanation hinges on the ratio between the speed at which productivity growth happens within-vs-across technologies. A higher $\bar{\lambda}$ indicates that the productivity growth rate within a technology is higher and can remain above the frontier growth rate γ for an extended period. During this period, as the vintage is becoming more productive relative to the just-born technology, its research share increases. As a result, increasing $\bar{\lambda}$ (everything else constant) shifts the peak of the distribution to the right.

Now consider Panel (b). The plots show the stationary quality distribution $Q\left(a\right)$ on the vertical axis comparing different values of $\bar{\lambda}$ and γ. Since quality is a cumulative process, the curves increase monotonically. Remarkably, they are S-shaped. Initially rather flat, the curves for quality then gain momentum and eventually flatten again. Observe how the distribution for a higher γ (green line) reaches its momentum earlier than in the baseline scenario (red line), consistently with the left plot in Panel (a).

Similar to Figure 3, Figure 4 presents illustrative calibrations for stationary distributions of the model, now for the measure of varieties (Panel a) and expenditure shares (Panel b). Again, in each plot, I show a baseline calibration in red, and two alternative calibrations in which I either increase the frontier growth rate γ (green curve), or the knowledge spillover parameter $\bar{\lambda}$ (blue curve).

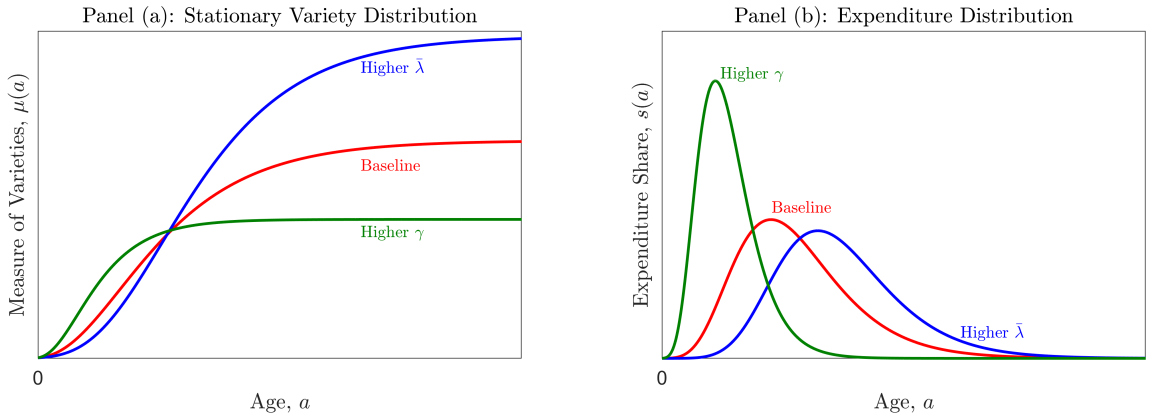

Figure 4: Cross-sectional technology-age stationary distributions for varieties and expenditure

Notes: The figure displays the stationary distributions of varieties $\mu(a)$ (Panel a) and expenditure shares $s(a)$ (Panel b) across technologies of different ages. Each plot shows the baseline calibration in red, the case of a higher frontier growth rate γ in green and the case of a higher step size on quality improvements $\bar{\lambda}$ in blue.

Observe how the patterns observed for the measure of varieties are similar to that exhibited by the quality distribution in Figure 3 (Panel a). As shown in (22), in the BGP, the measure of varieties moves in proportion to $Q(a)$. Finally, notice how the expenditure share distribution (Panel b), capturing the extent to which each technology is used in production, exhibits a distinct hump shape. Young technologies are not widespread because they are not yet perfected. Older technologies are abandoned because of the availability of better younger technology.

I now turn to the case where $\bar{\lambda} = 1$ for two reasons. First, while Figure 3 presents calibrations for which the distribution of research is hump-shaped, the case of $\bar{\lambda} = 1$ features an allocation of research that decreases monotonically with the age of a technology. Second, in this case, there is no growth in the quality ladder sense, and all improvements within a technology happen through the horizontal increase in the number of varieties.

Figure 5 presents the stationary distributions in the case of $\bar{\lambda} = 1$. In all plots, the horizontal axis represents the age of technologies. Panel (a) displays the research distribution $R(a)$ on the vertical axis, which monotonically declines with age. In fact, using Proposition 2 and the expression for research allocation derived in (13), it is straightforward to see that $R(a) \propto \exp(-a\gamma/\epsilon)$, which represents an exponential decay. In Panel (b), we see how the number of varieties $\mu(a)$ increases as a technology ages but without the S-shape distinguished pattern. The momentum of a technology happens right at its emergence, when the share of its researchers targeting it is maximum. The initial phase of the S-shape, when there is a slow accumulation of varieties preceding a period of higher growth, is thus not present. In Panel (c), the usage share of a technology presents a hump-shape, but the increasing part of the hump is not S-shaped either.

Despite the empirical relevance of the hump-shaped distribution of research and the S-shaped dynamics for the diffusion of technologies (see Section 2), the theory does not impose such results. These patterns are not replicated by construction, but rather inform the parameters of the model.

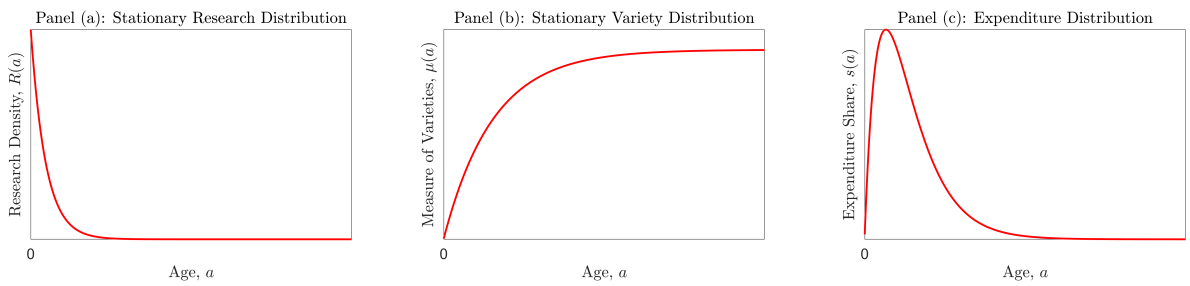

Figure 5: The cross-sectional stationary distributions with $\bar{\lambda} = 1$

Notes: The figure displays the stationary distributions of research $R(a)$ (Panel a), varieties $\mu(a)$ (Panel b), and expenditure shares $s(a)$ (Panel c) across technologies of different ages. In all plots, $\bar{\lambda} = 1$.

3.4 The Efficient Allocation of Research Across Vintages

In this section, I describe the efficient allocation of resources. The economy exhibits three standard sources of inefficiency commonly identified in the growth literature. The first is a static inefficiency arising from monopolistic competition in the production of various goods. The second is the duplication externality in the research sector, where multiple researchers may independently pursue the same idea. The third inefficiency stems from knowledge spillovers, as individual researchers do not consider how their discoveries affect the development of quality and variety within a technology, thereby influencing the value of future innovations.

Importantly, given the assumption that research is carried out by a separate group of workers whose supply is inelastic, the first two sources do not distort the allocation of researchers across technology vintages. A uniform change in the profitability of the monopolistic producers across technologies neither changes the total investment in innovation nor affects how it is distributed. Similarly, while the duplication externality might typically lead to over-investment in innovation, as noted by Jones (1995), this is not the case here due to the fixed supply of researchers. Furthermore, in the decentralized equilibrium, as discussed in Section 3.3, the expected marginal product of research[9] is equalized across technologies, leaving no room for gains from static reallocation. Knowledge spillovers, therefore, are the sole source of inefficiency that distinguishes the decentralized equilibrium from the efficient allocation of research across technologies.

[9]The expected marginal product of research at a technology τ is given by $\mathcal{I}(t|\tau)\bar{v}(t|\tau)$. That is, by the probability that an additional researcher in that field makes an innovation, $\mathcal{I}(t|\tau)$, times the expected value of an innovation, $\bar{v}(t|\tau)$.

The efficient allocation is characterized by the following optimization program:

$$\underset{\left[[R_t(a),\mu_t(a),Q_t(a)]_{a>0},C_t\right]_{t\geq t_0}}{\text{Max}} \int_{t_0}^{\infty} e^{-\rho(t-t_0)} \ln C_t \, dt \tag{23}$$

s.t.

$$C_t = Y_t - M_t = \tilde{\alpha} \int_0^{\infty} e^{-\gamma a} Q_t(a)\mu_t(a) da \tag{24}$$

$$\frac{\partial \mu_t(a)}{\partial t} = -\frac{\partial \mu_t(a)}{\partial a} + \eta R_t(a)^{1-\epsilon} \tag{25}$$

$$\frac{\partial Q_t(a)}{\partial t} = -\frac{\partial Q_t(a)}{\partial a} + Q_t(a)(\bar{\lambda}-1)\eta \frac{R_t(a)^{1-\epsilon}}{\mu_t(a)} \tag{26}$$

$$\text{and } R = \int_0^{\infty} R_t(a) da. \tag{27}$$

In this program, the objective function (23) represents the household's intertemporal utility. The resource constraints are given by (24) for the final good[10] and (27) for researchers. Finally, (25) and (26) delineate the innovation possibilities frontier.

Note in particular that the planner takes into account the effect of the research allocation across vintages on the law of motion of productivity within each technology, effectively internalizing knowledge spillovers. In standard models, knowledge externalities are homogeneous, so in general, there is no problem with 'which type of innovation' is relatively more desirable in social terms. In this model, however, the extent of the knowledge externality varies across vintages. The planner takes that into account. In particular, the planner realizes that there is a larger 'building on the shoulder giants' effect in younger technologies, something that is disregarded in the decentralized equilibrium.

To gain insight into the market inefficiencies and how they can be corrected, it is worth focusing on Balanced Growth Path (BGP) trajectories, in which consumption and output grow at constant rates, and the age distributions are time-invariant. Formally, I will consider the set with all feasible BGP trajectories (which contains, for instance, the decentralized equilibrium BGP previously derived) and look for the one that maximizes the household intertemporal utility.

Definition 3. (BGP feasible trajectories) A trajectory for the cross-sectional distributions of quality and varieties, $[\mathbf{\Theta}_t]_{t\geq t_0}$ (where $\mathbf{\Theta}_t \equiv [Q_t(a), \mu_t(a)]_{a=0}^{\infty}$), for the cross-sectional distribution of R&D, $[\mathbf{R}_t]_{t\geq t_0}$ (where $\mathbf{R}_t \equiv [R_t(a)]_{a=0}^{\infty}$), and for aggregate consumption $[C_t]_{t\geq t_0}$ is Balanced Growth feasible if, $\forall t \geq t_0, \forall a \geq 0$:

1.The distributions are time-invariant: $\partial_t Q_t(a) = \partial_t \mu_t(a) = \partial_t R_t(a) = 0$.

2.The resource constraint is respected: consumption is finite and satisfies (7).

[10]The second equality in (24) leverages the fact that the static allocation chosen by the planner only deviates from the decentralized equilibrium solution presented in (7) by a constant (in this case, we have $\tilde{\alpha}$ instead of α in (7)). This deviation occurs because the planner addresses the markup inefficiency. As noted, this inefficiency does not affect the planner's allocation of researchers since it uniformly impacts all vintages. We could set $\alpha = \tilde{\alpha}$ without affecting the subsequent results.

3. The total supply of scientists is respected: $\int R_t(a)da = R$.

4. The innovation possibilities frontier is respected: given the trajectory for $R_t(a)$, the trajectories for $Q_t(a)$ and $\mu_t(a)$ satisfy the laws of motion (17)-(18).

Corollary 3.1. In every feasible Balanced Growth trajectory, aggregate consumption grows at the constant rate γ. Namely, for every $t \geq t_0$:

$$C_t = A_t \hat{C}$$

where $A_t = e^{\gamma t}$, and $\hat{C}$ is a (trajectory specific) positive constant, which denotes the detrended level of consumption. The household intertemporal utility is hence $U = \int_{t_0}^{\infty} e^{-\rho(t-t_0)} \ln C_t dt = \frac{1}{\rho} \ln \hat{C} + B$, where B is a constant that depends on the parameters γ and ρ.

Proof. See subsection A.2 for the proof. □

Corollary 3.1 shows that the Balanced Growth feasible trajectory that maximizes the intertemporal utility of the household must be the one that delivers the maximum *level* of consumption, $\hat{C}$. This problem can be written as:

$$\underset{[R(a)]_{a>0}}{\text{Max}} \ \kappa \int_0^{\infty} e^{-\gamma a} Q(a) R(a)^{1-\epsilon} da, \tag{28}$$

s.t.

$$Q'(a) = Q(a)(\bar{\lambda} - 1)\frac{\eta}{1-\epsilon}\frac{R(a)^{1-\epsilon}}{\mu(a)}, \tag{29}$$

$$\mu'(a) = \frac{\eta}{1-\epsilon} R(a)^{1-\epsilon}, \tag{30}$$

$$\text{and } R = \int_0^{\infty} R(a) da. \tag{31}$$

where (28) represents the level of consumption $\hat{C}$ being maximized. It was obtained from the resource constraint (24) and transformed through integration by parts[11]. In turn, (29) and (30) represent the laws of motion for $Q_t(a)$ and $\mu_t(a)$ after imposing the condition $\partial_t Q_t(a) = \partial_t \mu_t(a) = \partial_t R_t(a) = 0$.

The objective function (28) shows the direct impact of allocating $R(a)$ researchers to the age a technology. This yields an innovation flow rate equal to $\eta R(a)^{1-\epsilon}$, leading to the creation of varieties whose average quality is proportional to $Q(a)$, and whose intrinsic potential relative to the frontier trend is $e^{-\gamma a}$. In turn, the law of motion in (29) captures the impact of innovation on the average quality evolution. The more innovations are made in a technology of age a, the higher will be its average quality $Q(a')$ at older ages $a' > a$, and therefore the greater the direct impact of research in these later stages, $e^{-\gamma a'} Q(a') R(a')^{1-\epsilon}$. This represents a classic standing on

[11] The expression in (28) equals the consumption level $\hat{C}$ up to a positive and finite additive constant (which does not distort optimality conditions).

the shoulder of giants' externality, and I will hereafter refer to it as such, or, for brevity, as the *Shoulder's externality ($\mathcal{E}_S$)*. It is captured by constraint (29) in the above problem.

One of the reasons this way of writing the problem is helpful is that $\mu(a)$ does not appear directly in the objective function, only in the constraints. This makes it clear how innovation $R(a)$, although having a direct positive effect on aggregate consumption (28), also has a cost, making ideas harder to find. It increases $\mu(a)$, as seen in equation (30), which in turn will make proportional improvements on the average quality more difficult: in (29), the higher $\mu(a)$, the smaller $Q'(a)/Q(a)$ is in the face of a flow of research $R(a)$. This is the second externality not taken into account in the decentralized equilibrium, which I will refer to as the *Dilution externality ($\mathcal{E}_D$)*. It is captured by constraint (30) in the problem above.

Proposition 3. *Within the set of feasible BGP trajectories (see Definition 3), consider the problem of choosing the one that maximizes the household intertemporal utility. In a solution, the allocation of innovation must satisfy, for every $a \geq 0$:*

$$R(a)^{\epsilon} \propto \underbrace{e^{-\gamma a}Q(a)}_{\substack{decentralized\\ equilibrium}} \kappa + \underbrace{\psi_Q(a)\eta(\bar{\lambda}-1)\frac{Q(a)}{\mu(a)}}_{\mathcal{E}_S(a)} + \underbrace{\psi_\mu(a)\eta}_{\mathcal{E}_D(a)} \tag{32}$$

where $\psi_Q(a)$ and $\psi_\mu(a)$ are the costate functions associated, respectively, with the constraints (29) and (30).

If $\bar{\lambda} = 1$, there are no externalities, $\mathcal{E}_S(a) = \mathcal{E}_D(a) = 0$ for every $a \geq 0$, and the decentralized equilibrium allocation of research is efficient in the BGP.

If $\bar{\lambda} > 1$, the two externalities are active and satisfy:

1. *(Building on the shoulder of giants) For every $a \geq 0$:*

$$\mathcal{E}_S(a) > 0 \text{ and } \mathcal{E}_S'(a) < 0 \tag{33}$$

The building on the shoulder of giants' externality leads a benevolent planner to innovate relatively more on young technologies compared to the decentralized equilibrium.

2. *(Dilution effect) For every $a \geq 0$:*

$$\mathcal{E}_D(a) < 0 \text{ and } \mathcal{E}_D'(a) > 0 \tag{34}$$

The second externality (the dilution effect) leads a benevolent planner to innovate relatively less on young technologies compared to the decentralized equilibrium.

*Proof.*See subsection A.3 for the proof. □

Proposition 3 contains three important results concerning the efficient allocation of research in the BGP. First, it shows in (32) how the two externalities highlighted above, Shoulders and

Dilution, are the forces deviating the decentralized equilibrium from the efficient allocation. Were they not active, the equilibrium allocation of research would be efficient. This occurs if $\bar{\lambda} = 1$, and hence $Q(t|\tau)$ is constant in time, indicating the absence of any perfection process within each technology. In this case, the benefits of research are limited to the horizontal expansion in the mass of varieties, with its direct effects to aggregate consumption fully internalized in private profits. If $\bar{\lambda} > 1$, however, knowledge accumulates within a technology in the form of its perfection process. By taking into account the externalities involved in this process (Shoulders and Dilution), a benevolent planner can lead to a greater accumulation of quality within each vintage, and hence to a higher level of detrended consumption.

It is, therefore, crucial to understand how these two externalities shape the optimal allocation of research in the case of $\bar{\lambda} > 1$. The third important result of Proposition 3, contained in (33), does so for the former. Proposition 3 shows that although the building on the shoulder of giants' externality results in a benevolent planner valuing innovation more than the market within every technology ($\mathcal{E}_S(a) > 0$ for every a), it values it *relatively* more than the market for younger technologies ($\mathcal{E}'_S(a) < 0$ for every a). In other words, the distribution of research efforts is biased towards old technologies in the decentralized economy - as far as the building on the shoulder of giants' effect is concerned.

Finally, the fourth important result from Proposition 3 relates to the second externality, the Dilution effect $\mathcal{E}_D$. In contrast to the first one, it leads the planner to value innovation less than individual researchers, and it values it relatively less for young technologies. This is intuitive: early inventions will make the average quality of a field less elastic than the contribution of older inventions. By internalizing this cost, a benevolent planner would be less restrictive on innovating in older technologies than younger ones. For example, the cost of a bad idea outcome is much smaller when the technology is old and many inventions have already been made: the new one will have only a marginal influence on the average quality.

The net effect of externalities $\mathcal{E}_S(a)$ and $\mathcal{E}_D(a)$ for a given technology a depends on the parameters. It can be shown, for example, that when $\bar{\lambda}$ is sufficiently high, the building on the shoulders of giants' externality will dominate, in the sense that $\mathcal{E}_S(a) + \mathcal{E}_D(a) > 0$. Regardless of the net signal of such effects, however, an important additional result regarding the efficient allocation can be derived, which is the subject of Proposition 4.

Proposition 4. *Within the set of feasible BGP trajectories, consider the problem of choosing the one that maximizes the household intertemporal utility. In a solution, the allocation of innovation must satisfy for every $a \geq 0$:*

$$R'(a) < 0 \text{ and } \lim_{a \to \infty} R(a) = 0$$

The optimal allocation of research is not hump-shaped. Instead, research effort monotonically falls with age.

*Proof.*See subsection A.3 for the proof. □

Proposition 4 states that the planner distribution of research is decreasing across vintages ($R'(a) < 0$): older technologies receive less weight and are abandoned asymptotically ($\lim_{a\to\infty} R(a) = 0$). This is in contrast to the decentralized equilibrium inverse U-shaped allocation of scientists. When a technology emerges with little accumulated knowledge (low perfection level or average quality), there is value for individual scientists in waiting until others develop the foundations of the new field, so that they can jump in later, when returns in older fields are small and the knowledge accumulated in the new one is higher. This waiting value is not present for a planner, who prefers to develop the new field as soon as possible.

Discussion

It is well known in the growth literature that the building on the shoulder of giants' externality leads to underinvestment in innovation, as inventors' profits do not incorporate positive spillovers on future research in a market equilibrium. This brings no surprise to the result that $\mathcal{E}_S(a) > 0$ for every $a \geq 0$, meaning that the planner values more innovation in every technology. The novel mechanism uncovered by this theory, however, is that in an economy where individual technologies eventually lose momentum and are gradually replaced by newer vintages, the non-internalized spillovers of research are higher for young than for older technology, $\mathcal{E}'_S(a) < 0$. In contrast to the literature on creative destruction following Aghion and Howitt (1992), which considers the accumulation of knowledge to be uniform, I assume there are different blocks of knowledge (technologies) that gradually and perpetually replace each other, leading to the emergence of an intrinsic inefficiency in the allocation of innovation. Of course, depending on the theory one has in hand, other externality channels may be present in addition to this one, making the final results regarding underinvestment in innovation ambiguous. This is the case of the business stealing effect in the creative destruction literature (Aghion and Howitt, 1992), and will be the case of ideas getting harder within vintage in this paper. Nonetheless, the widespread presence of the standing on giants' shoulders externality on growth models, either alone or combined with others, makes the result in Proposition 3 important.

To conclude, reallocating research cannot affect the growth rate of the economy in the long run, but by internalizing the spillovers and influence current research within a technology has on future innovations, it can lead to a great accumulation of quality throughout a technology life cycle. These higher levels of perfection achieved by each technology lead the economy to a higher level of consumption and output compared to the decentralized equilibrium.

4 Measurement: Data and Technology Vintages

In this Section, I describe the patent data used in the quantitative part of the paper and how I measure the vintage structure of the model in the data, taking as a reference the technological classification of the US Patent Office. This includes a detailed discussion of the methods and results presented in Section 2 related to the date of emergence of technologies and the hump shape

in research allocation.

4.1 Data: patents and technologies

My dataset comprises approximately nine million patents, constituting virtually the universe of utility patents issued by the United States Patent Office (USPTO) from 1836 to 2010. The data is collected from the USPTO Historical Patent Data Files (Marco et al., 2015). Unless otherwise noticed, patents are binned by decade given their issue date. Although the application date might provide a better view of when the invention was done, this information is scarcely available for old patents. Berkes (2016), however, show that the average time necessary to issue a patent was less than one year in the nineteenth century, and between two and three years in the 20th century, a small magnitude compared to the long-term perspective adopted in this analysis. Panel (a) in Figure 6 shows that the number of patents issued per decade grew over time, from less than 6,000 in the 1840s to more than 1.5 million in the 2000s.

Information on the market value of patents comes from Kogan et al. (2017). Using stock markets' reaction to news about patents, the authors estimate the valuation of $1,928,123$ patents assigned to publicly traded firms in the period 1926-2010. To control for patent quality, I again use Kelly et al. (2021)'s importance index, available for virtually the universe of patents issued in the period 1836-2010. As discussed, the availability and relevance of citation data are restricted when looking into long-term time horizons. Not only the recording of citations by the USPTO is relatively recent (1947), but most importantly, the institutional standards and parameters governing the legal necessity and the conditions under which citations were made changed a lot over time. Kelly et al. (2021) overcome these shortcomings by using a textual approach, equally applicable to all patents in the records. In accordance with their methodology, the importance of a patent is higher the more dissimilar it is from the prior art, and the more similar it is to the following inventions, i.e., the more novel and influential it is. The authors validate their index with many measures for quality, showing its correlation with citations and patent value (when observed).

Technology classes

A key aspect of my theory is the technological vintage structure. To operationalize this concept in practice, I rely on the technological classification of the USPTO. I use the US patent classification (USPC) which is structured under the principle that "a class generally delineates one technology from another" and that "classes are mutually exclusive, meaning that the subject matter provided for by one class does not overlap that provided for by another" (USPTO, 2012). I hence can define each of the 401 USPC utility classes[12] to represent a technology in the analysis that follows.

It is important to note that the classification attributed to each patent in my dataset, from the older to the newest, is consistent with the latest version of the USPC code. Every time a class is

[12] As implied, I exclude Design classes as well as Plant classes, keeping only the Utility ones. I also retain only Original Classifications (OR) classes, representing the main technologies to which a patent can be linked to.

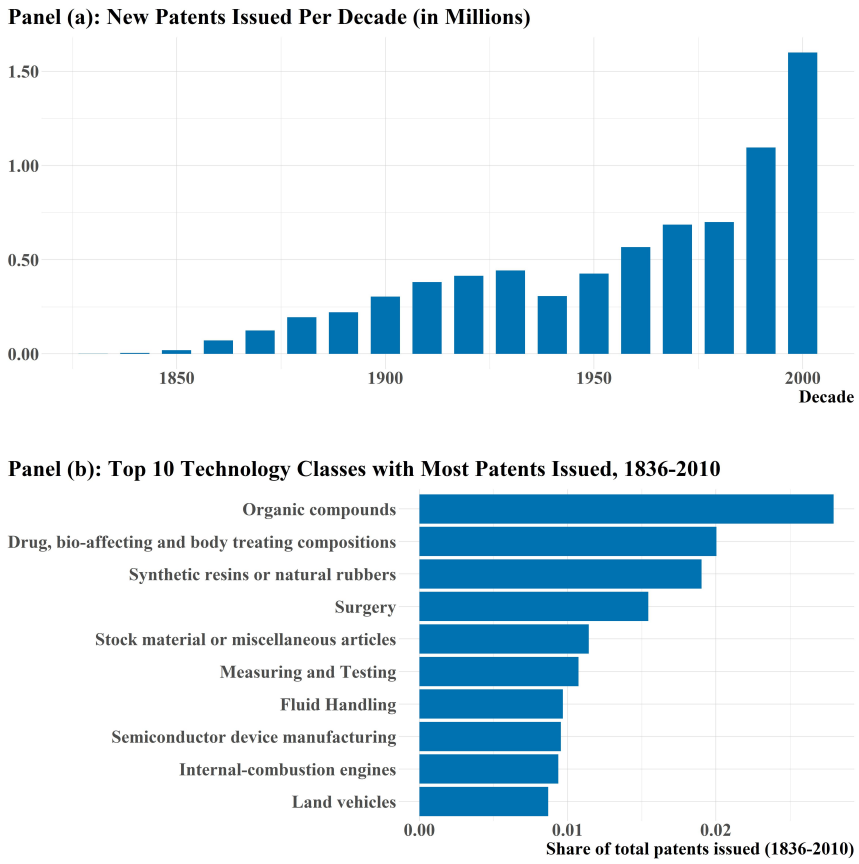

Figure 6: Issued Patents Over Time and Main Technology Classes

Notes: The data in both panels comes from the USPTO Historical Patent Data Files (Marco et al., 2015). Panel (a) depicts the total number of utility patents issued by the USPTO per decade throughout the period 1836-2010. It excludes patents whose issue date or classification code is missing, as well as patents considered as "Unclassified" (USPC class 1) by the Patent Office. Panel (b) depicts the 10 technology classes in the USPC code that encompass the largest number of patents in the period, with the respective share of total patents. When computing this, I only consider the primary classification, so each patent is matched to one and only one technology class.

created or abolished, the USPTO retroactively reclassifies all issued patents until that moment, so that the classification of all of them becomes consistent with the new version of the code. The 401 USPC technologies used here are present in its latest version, and all patents in my dataset are classified according to it. Furthermore, I focus only on the primary classification of each patent, meaning that each patent is classified into one and only one technology.

Most patent classes in the USPC are structured under the principle of proximate function: they group inventions that perform one specific function, one fundamental task, regardless of the sector or type of application that each invention may be linked (USPTO, 2012; Lafond and Kim, 2019). For instance, a butter churn and a mixer (shaken) are classified together in the class 'Agitating' even though having different applications (USPTO, 2012). This means that the concept of technology in this empirical application should be understood as a body of knowledge to perform one particular

task or function. Independently of that, most important for this paper, technological classes inherit the main feature of a technology in my theory: a cluster of knowledge, expanded by cumulative inventions which take as an input the cluster's knowledge more than any other technology. Liu and Ma (2023) show that patents within a class are mostly likely to cite patents within that same class. Acemoglu et al. (2016b) also presents strong evidence on this (although their unit of analysis was a level of aggregation above the individual classes). Érdi et al. (2013) used cluster analysis with citation network and showed that this could predict the creation of new technological classes in the patent classification.

Panel (b) in Figure 6 and Table1 present important examples of technologies in the USPC classification code. In the former, one can see the 10 technology classes to which most patents were assigned throughout the entire sample period (1836-2010) - and the share of patents accrued by each. Biology and chemistry technologies play a prominent role, followed closely by the popular cases of the internal combustion engine and semiconductor devices. Other technologies with broad applicability, such as measuring and testing, or fluid handling, are also important. In Table 1, some selected key technologies in the process of growth are displayed, as railways, software development, and artificial intelligence. The shares of patents accrued by them in 1950 and 2000 can also be seen in the two last columns.

4.2 Measuring the emergence of new technologies

In my theory, a vintage or technology is defined by the date of its emergence. Hence, I need a procedure to date technologies, establishing how old or new they are at a specific time. For this purpose, as first discussed in Section 2, I follow the seminal paper by Griliches (1957). The idea here is that technologies diffuse in the innovation space following a qualitative behavior similar to the one observed in the product and consumption space. By the latter, I refer to the gradual increase in the share of firms and consumers adopting a given technology, as was the focus of Griliches and much of the diffusion literature. By the former, I refer to the gradual expansion in the share of innovations made related to a given technology. More and more, it starts to be adopted by researchers and R&D labs as a platform in which they can build. Beginning from low levels both in absolute and relative terms, patenting activity within a technology grows over time, reaching a peak and later declining. This S-shape diffusion behavior lends itself naturally to a logistic fit, as explored by Griliches (1957).

This intuition is supported by the example of individual technologies, as the steam engine case in Figure 1, or by the technology classes of the USPC system which are the focus here. For instance, notice in Panel (a) of Figure 7 how the share of patents for 'Amplifiers' is small and only slowly increasing by the turn of the 19th and 20th centuries. It then gained momentum and significantly increased before reaching a peak in 1960, a trend also observed in the other panels of Figure 7.

Following Griliches (1990), I run a logistic regression for each technology class, so as to fit its diffusion trend in the innovation space. Specifically, for each class, I define its diffusion period as the period that ends when its share reaches the maximum. I then fit a logistic curve to it and use

the 10 percent of such maximum value as a crossing point to define the origin date proxy. Similarly to the notion of availability in Griliches, the focus is not on the first discovery associated with a technology, but on identifying when it becomes a significant platform for subsequent innovations.

Figure 7 further clarifies this procedure by providing examples of the estimation. To begin, first notice that the dots in Panels (a)-(d) depict the share of patents accrued by each technology over time - and have been normalized by the maximum value achieved in each case. For instance, the share of patents associated with the 'Amplifiers' technology among all patents issued in a given decade reached a maximum in the 1960s, being normalized to unit by then. This maximum value also delimits the sample used for the fit: only a technology's expansion (diffusion) period in the innovation space is used, which I define as the entire time period preceding the maximum period. The resulting logistic trend for the diffusion of each technology is represented by the solid line. Finally, the doted lines indicate the point (and the year) in which the estimated trend reached 10% of its maximum value. For the 'Amplifiers' technology, this happened in 1911.

Second, observe in Figure 7 the possibility of extrapolating the date of origin into the pre-sample period, which becomes clear when comparing Panels (a) and (b). In the 'Amplifiers' case, the first observations of its patent share are relatively far from the peak value, slowly increasing at first and then accelerating to approach the peak. This is the typical start of the diffusion pattern documented extensively in the S-shape literature. As a result, the logistic fit interprets the first observations as, in fact, the start of the technology diffusion process. In the 'Coating' technology case, however, when the records of patents from the US patent system started, at around 1830, the patenting share in Coating was already relatively stable and close to its peak. The logistic fit interprets this as an indication that the technology had emerged further in the past, having already surpassed the early stages of its diffusion period among inventors in previous decades. As a result, the logistic fit extrapolates the 10 percent mark to the year 1779.

From a total of 401 technology classes, the origin date was estimated by the above method for 343 of them. This is because, for some classes, the maximum achieved patenting share is such that no more than one observation remains for the logistic estimation (observations for patents are binned into 10-year periods). Additionally, to remove outlier estimates projecting the origin of technologies far back in the past or ahead in the future, the results were winsorized at the 2.5 and 97.5 percentiles, leaving me with 325 classes. They represented 93% of all patents issued in the 2000 decade and 76% in 1900. In the empirical analysis that follows, I will also consider for robustness a more selective group with 238 classes for which the R square in the logistic estimation exceeds 0.5.

Crucially, this empirical approach does not impose whether the share of innovation in a given technology, which is, for example, 1 year old, will be higher or lower than the share in a 50-year-old one. Similarly, it does not impose that two technologies will have the same patent share when each of them has a given age, say 10 years old. When establishing the 10 percent cutoff, the date of origin estimation takes as a reference the maximum share achieved by each technology individually, without direct relation to others. Take as an example the cases of technologies 'Tuners', represented

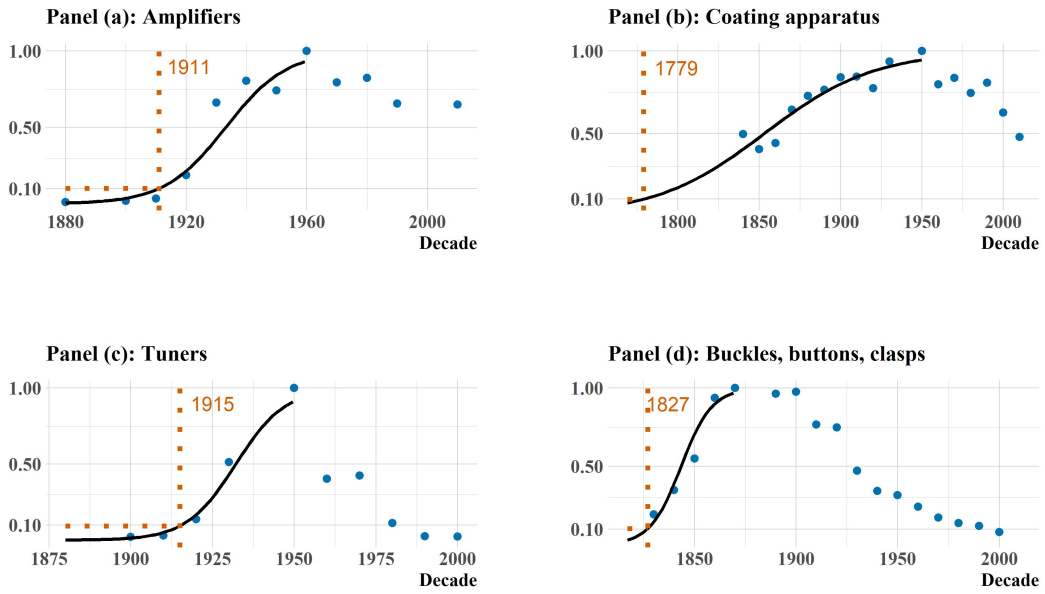

Figure 7: Patenting share and proxied emergence date

Notes: The dots in each figure represent the share of patents accrued by the respective technology class among all utility patents issued in a given decade. For each technology, observations below the 5th and above the 95th percent data were removed to avoid the influence of outliers. Shares are normalized in each figure by the maximum value achieved. The solid line represents the fitted logistic trend whose estimation is described in the text of this section and in Appendix B.4. The dashed lines represent the proxied date of origin: the year in which the trend achieved 10% of the ceiling value.

in Panel (c) of Figure 7, and 'Pipe Joints' (Figure C.8 also presents the evolution in the share of patents of both technologies with the actual, i.e., non-normalized values). The highest innovation share achieved by 'Tuners' was in 1950 and it is less than five times the value for the earliest observation of Pipe Joints. The fact that a technology is young or old does not imply mechanically that its innovation share is higher or lower than others, at a fixed moment in time or at different points of their life cycle.

Results

The second column in Table 1 presents the estimated emergence date for a set of key technologies. For instance, the 'Television' technology was dated to 1929. Reassuringly, although broadly defined experiments related to image wave transmission have been taking place since the 19th century, practical electronic television systems that resemble modern television were developed in the 1920s and 1930s[13]. It is also possible to see in the last two columns how the share of patents associated with 'Television' more than doubled between the 1950s (when the technology was new) and the

[13]For example, Philo Farnsworth conducted the first successful electronic television demonstration on September 7, 1927, laying out the foundation for the modern television system that uses electronic scanning to produce images.

Table 1: The emergence of key technologies

Technology class	Origin	Patenting share (%)	
		1950s	2000s
Gas: heating and illumination	1766	0.14	0.02
Aeronautics	1871	0.55	0.22
Railway draft appliances	1826	0.10	0.00
Electric lamp	1878	0.06	0.08
Television	1929	0.36	0.81
Semiconductor device manufacturing	1964	0.05	3.07
Data Processing: artificial intelligence	1968	0.00	0.15
Software development, and installation	1985	0.00	0.34
Information security	1987	0.00	0.25

Notes: This table presents examples of technology classes (USPC classification system) with their corresponding estimated date of origin and patenting shares in two different decades. Such shares correspond to the total number of patents assigned by the US Patent Office to the class (primary classification), divided by the total number of patents issued during the decade. The date of origin is estimated with a logistic regression, following the approach in Griliches (1957), as described in the text of this section and in Appendix B.4.

2000s. Turning now to a technology whose emergence date was extrapolated to the pre-sample period (i.e., before the first issued and available patents), Table 1 shows the case of 'Gas: heating and illumination'. The estimated date, 1766, lies in the second half of the 18th century when experiments with gas for illumination started to gain prominence, as illustrated by the famous breakthroughs from William Murdoch. Finally, Table 1 provides examples of digital and information technologies whose emergence was estimated to the second half of the 20th century.

More generally, Figure 8 shows a time series with the number of technologies emerging in each decade. Three waves can be distinguished, representing periods with many technologies being created. The first wave happened early in the 19th century, coinciding with the first industrial revolution in the United States. In my empirical analysis, it represented the emergence of several technologies serving as a base for general manufacturing activities, such as 'Endless belt power transmission system', and 'Turning'. It also marked the emergence of the first classes associated with railways and electricity. The second wave can be distinguished late in the XIX century, with a peak during the 1880s, and expanded the period until the 1910s. It is dominated by the emergence of several classes related to electricity and the chemical industry, such as 'Battery or capacitor charging or discharging' and 'Synthetic resins or natural rubbers', coinciding with the period of the second industrial revolution. The third wave happened in the second half of the 20th century, with its peak in the 1960s. Within it, several technologies related to computing and telecommunications were estimated to emerge, as is the case of 'Data Processing: artificial intelligence', in 1968[14].

[14]For comparison, the origin of AI is often attributed to the seminal conference in 1956 in which the term was first used: the Dartmouth Summer Research Project on Artificial Intelligence (Russell and Norvig, 2009).

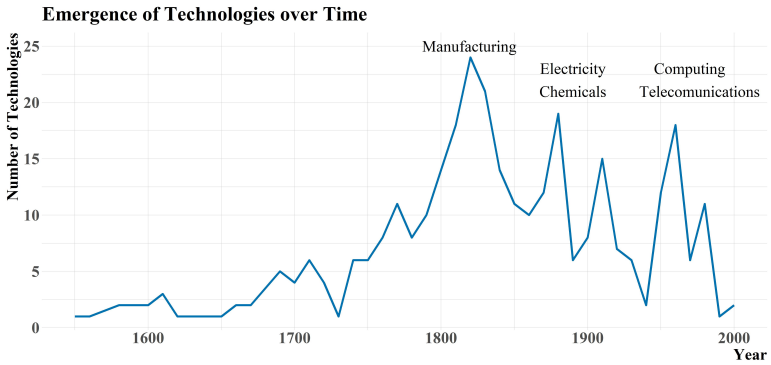

Figure 8: The emergence of technologies over time

Notes: The figure shows the number of technology classes (UPSC classification system) that emerged in each decade. The emergence decade is estimated with a logistic regression, following the approach in Griliches (1957), as described in the text of this section and in Appendix B.4.

The Hump-Shape of US innovation After defining technologies, assigning patents to each of them, and estimating their emergence date, I can study the allocation of innovation between technologies of different ages and how it changed over time. The main results were discussed in Section 2. Figure 2 presented the share of patents, at a given point in time, accrued by new and old technologies, from which a pronounced hump shape over technology age can be distinguished. Moreover, the comparison of the distribution in 1900 and 2000 showed a remarkable stationary behavior in the distribution.

Notably, Figure 9 shows that this stationary pattern is not exclusive to the years 1900 and 2000 but extends across all decades of the 20th century. For each decade, the figure depicts the distribution of patents across technologies of different ages, as measured by the share of patents accrued by each of these classes. Both the prominent hump-shaped trend and the distribution's peak, concentrated around technologies aged between 70 and 100 years, remained stable throughout the century.

Additional robustness graphs are contained in Appendix C. In particular, Figures C.3 and C.6 consider the patent share of technologies not among all US patents, as has been the case in Figure 2, but among the patents of the 325 classes whose date of origin has been dated by the logistic estimation above. As discussed, they correspond to 93% of all US patents issued in 2000 and to 76% in 1900. For a given period (say, the 2000 decade), this does not affect the hump shape, representing a normalization that shifts the curve's position upward. When comparing the distribution in different periods, it is possible to see a vertical increase in the difference between the 1900 and 2000 curves, as the normalization factor in the former year is larger than in the latter. This, however, does not affect the peak of both curves, which remains quite close at approximately one hundred years, nor their shapes and overall proximity. The notion of stationarity remains quite striking. Finally, Figures C.4 and C.7 consider a more selected set of vintages for which the confidence in the date of origin estimates is higher. These vintages result from the 238 technologies

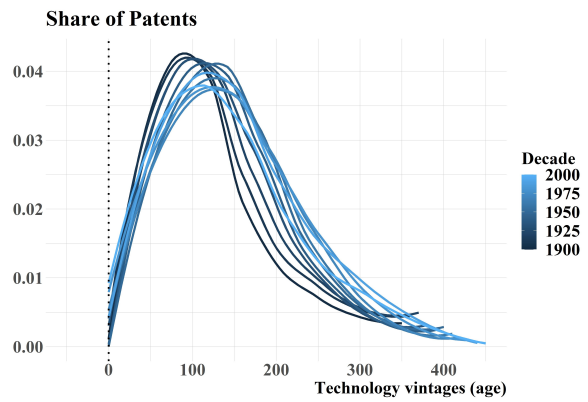

Figure 9: The Hump-Shaped Innovation-Age Profile

Notes: For each decade of the 20th century, the figure shows the distribution of patents sorted according to the age of the technology class they are associated with. Each line corresponds to a non-linear fit of the distribution for the respective decade. The age of each technology is estimated in accordance with Griliches (1957) methodology (see the text of this Section).

whose logistic estimation in Section has R square greater than 0.5. Once again, results regarding the inverse U-shape in the allocation of research and its stability are persistent.

5 Calibration

To quantify the potential welfare losses stemming from research misallocation, I calibrate the structural parameters of my model. To this end, in this Section, I match the model BGP equilibrium to the US economy in the 2000 decade. Focusing on BGP is motivated by the stationary behavior in the allocation of research observed in the data.

5.1 Taking the model to the data

I assume that the observed number of patents is an indicator, albeit noisy, of the total number of new ideas generated in a specific time period. Formally,

$$Patents(t|\tau) = \kappa_\tau \times \dot{\mu}(t|\tau) \times u(t|\tau), \tag{35}$$

where $Patents(t|\tau)$ denotes the observed number of patents issued in a given period within the technological class τ, while $\dot{\mu}(t|\tau)$ is the true number of new ideas in my model. In turn, κ_τ is a constant unique to the technological class, indicative of its patenting propensity, and $u(t|\tau)$ is a scalar disturbance term.

The link established by Equation (35) between the model and the data presents two clear advantages. First, it doesn't presuppose that every novel idea culminates in a patent, sidestepping a widely acknowledged constraint when leveraging patent data to gauge innovation (Griliches, 1990;

Hall et al., 2001). Specifically, Equation (35) implies that on average only a fraction κ_τ of all ideas within technology τ are patented. Second, it accommodates varying patenting propensities across diverse technologies. This flexibility is crucial, especially when examining patents spanning an array of technologies and sectors (Hall et al., 2001).

I now use the model equilibrium conditions to substitute for $\dot{\mu}(t|\tau)$ in Equation (35). In the model, by definition, the flow of new ideas is related to the aggregate number of researchers working on technology τ, $R(t|\tau)$. In turn, using (13) and (9), I can relate this research effort to the market value of patents in τ. Using this information on (35), gives the regression equation

$$\log Patents(t|\tau) = \delta_\tau + \delta_t + \frac{1-\epsilon}{\epsilon} \log \bar{v}(t|\tau) + \log u(t|\tau). \tag{36}$$

Here, $\bar{v}(t|\tau)$ denotes the average market value of patents issued in period t pertaining to technology τ. The terms δ_τ and δ_t represent technology and time fixed effects, respectively. These terms encapsulate κ_τ, as well as parameters and aggregate variables from the model.[15] Notice in (36) that a high ϵ means that research supply is relatively inelastic: substantial variations in patent valuation lead to only minor changes in the corresponding research efforts. I will utilize this observed variation to determine ϵ.

To be able to consistently estimate ϵ in the regression implied by (36), I assume that the disturbance term $u(t|\tau)$ accepts the representation $u(t|\tau) = \alpha_\tau \epsilon(t|\tau)$. Here α_τ is a time-invariant term that potentially correlates with $\bar{v}(t|\tau)$ across τ. In turn, $\epsilon(t|\tau)$ must be uncorrelated with $\bar{v}(t|\tau)$. The identification assumption, therefore, is that if the average value for ideas (considering both patented and non-patented ones) in a given technology is abnormally high for a technology τ in a given year t, then it cannot be true that researchers who target τ would patent more (or less) relative to the actual innovations $\dot{\mu}(t|\tau)$ in that particular year.

Table 2 reports the results. In columns (1)-(3) technologies correspond to patent classes in the USPC technological classification code. Meanwhile, in columns (4)-(6) technology classes are grouped by age, representing technology vintages, following the estimations presented in Section 4. The estimation sample is restricted to patents assigned to publicly traded firms since these are the ones for which I can backup the independent variable $\bar{v}(t|\tau)$ using Kogan et al. (2017) dataset (see Section 4). In columns (1)-(3), I trim the bottom 2.5% of both patent values and patent flows to reduce the influence of outliers. However, for columns (4)-(6), given the small sample size when we cluster technologies by their emergence date, I refrain from doing it.

The findings presented in Table 2 provide estimates for ϵ that range between 0.75 and 0.89 when incorporating both technology and time fixed effects, in line with the full specification in Equation

[15] Specifically,

$$\delta_\tau = \log \kappa_\tau + \log \eta - \frac{1-\epsilon}{\epsilon} \log(\beta \bar{\lambda})$$
$$\delta_t = -\frac{1-\epsilon}{\epsilon} \gamma t - \frac{1-\epsilon}{\epsilon} \log(D_t \bar{Q}_t).$$

Table 2: Estimates for the research supply elasticity ϵ

	Number of new patents: $\log Patents(t\|\tau)$					
	τ = USPC patent classes			τ = Technology vintages		
	(1)	(2)	(3)	(4)	(5)	(6)
Avg Patent Value $\log \bar{v}(t\|\tau)$	0.368	0.128	0.115	0.639	0.326	0.317
	(0.089)	(0.043)	(0.042)	(0.226)	(0.090)	(0.088)
	$\epsilon = 0.731$	$\epsilon = 0.887$	$\epsilon = 0.897$	$\epsilon = 0.610$	$\epsilon = 0.754$	$\epsilon = 0.759$
Year FE	✓	✓	✓	✓	✓	✓
Tech FE		✓	✓		✓	✓
N	1044	1044	1044	168	168	168
F-Stat	17.329	8.75043	7.379	8.019	12.962	12.977

Notes: This table provides estimates of the parameter ϵ based on the regression detailed in (36). The dependent variable represents the logarithm of patent counts, while the explanatory variable corresponds to their average market valuation. The data on patents and valuations for these regressions comes from Kogan et al. (2017). For columns (1)-(3), technologies correspond to classes in the USPC classification code, while in (4)-(6) they correspond to vintages grouped by emergence date, as detailed in Section 4. In columns (3) and (6), each patent is weighted by its importance indicator estimated by Kelly et al. (2021). The dataset spans 1990-2004, with data aggregated in 5-year intervals.

(36). In the absence of technology fixed effects, ϵ estimates drop to a minimum of 0.61 in column (4). Furthermore, when considering patent quality, the ϵ estimates remain stable. Specifically, in columns (3) and (6), I use a quality-weighted sum of patents, drawing on Kelly et al. (2021)'s patent importance metric, in place of the sheer patent counts. These variations do not affect the outcomes.

In the subsequent quantitative analysis of the model, I will refer to various ϵ values from Table 2. However, for the baseline calibration, I will adopt an intermediate value from Table 2, specifically $\epsilon = 0.75$, as estimated in columns (5) and (6).

Calibrating the frontier growth rate parameter γ

Another key parameter in the model is γ, representing the rate at which new technologies become intrinsically better. As shown in Proposition 2, along a balanced growth path, output per capita grows at rate γ. I therefore set $\gamma = 2\%$, which has been approximately the average growth rate of GDP per capita in the US economy for the last century.

Calibrating the knowledge spillover parameter $\bar{\lambda}$

I next focus on the parameter $\bar{\lambda}$, which represents the step size in quality of newly issued patents relative to the average of existing ideas. As profits in my model are proportional to the quality of an idea, $\bar{\lambda}$ represents the average increase in the profitability of successive generations of varieties within a technology as it ages. Motivated by this, I exploit the age profile of patent valuations to inform the calibration of $\bar{\lambda}$.

In particular, to describe how valuation changes as time passes and technologies age, I compute the average growth rate of $\bar{v}(t|\tau)$ over time, $d\mathrm{E}_\tau \log \bar{v}(t|\tau)/dt$. To estimate it, I regress $\log \bar{v}(t|\tau)$ on

Table 3: Average Growth Rate in Patent Value across technologies

	Avg Patent Value, $\log \bar{v}(t\|\tau)$	
	(1)	(2)
Time trend, t	0.089	0.088
	(0.004)	(0.004)
Tech FE	✓	✓
N	686	672
Adj R^2	0.47	0.55

Notes: The table presents the fixed effect Within estimator for a regression of $\log \bar{v}(t|\tau)$ on a time trend. The data ranges from 1990-2004 and is binned in 5-year periods. Outliers are removed in (2), by truncating the top and bottom percentiles of patent values. Technologies τ correspond to classes in the USPC classification code, grouped in vintages as described in Section 4.

time t, as reported in Table 3. The estimated coefficient represents the partial effect of time on the valuation of newly issued patents, $\log \bar{v}(t|\tau)$, averaging across technologies. Columns (1) and (2) present the results when controlling for technology fixed effects. On average across technologies, the market value of new patents increases at an yearly rate of 9% as technologies age.

To discipline $\bar{\lambda}$ with this moment, we can derive its counterpart in the model. Using Equation (9) defining $\bar{v}(t|\tau)$, it is possible to write:

$$\frac{d}{dt}\mathrm{E}_\tau \log \bar{v}(t|\tau) = \frac{d}{dt}\log D_t + \frac{d}{dt}\mathrm{E}_\tau \log Q(t|\tau) \tag{37}$$

where, as defined in (8), $D_t \equiv \int_t^\infty \exp\left(-\int_t^s r(v)dv\right) ds$. Moreover, the growth rate in D_t is equivalent to the growth rate in the stock market to GDP ratio (see Appendix B.3 for the derivation). Equation (37) shows us that over time, newly issued patents within a technology become more valuable for two reasons. First, due to an aggregate effect that globally increases the value of patents and is reflected in the variation of the stock market to GDP ratio. Second, because the technology's own stock of knowledge is increasing: its new patents are each time more productive.

Therefore, with our empirical estimate for the growth of $\bar{v}(t|\tau)$, corresponding to the left side of the Equation (37), if we subtract from it the growth in the stock market to GDP ratio, also observed empirically, we obtain a measurement for the growth in $Q(t|\tau)$, which is highly informative of $\bar{\lambda}$. My calibration strategy consists of choosing $\bar{\lambda}$ so that the simulated model can replicate this moment.

I compute the yearly growth rate for the stock market to GDP ratio using the FRED databse.[16] I obtain a yearly growth rate equal to 0.08712. Therefore, using the estimate from Column (1) in

[16]Time series 'Stock Market Capitalization to GDP for United States, Percent, Annual, Not Seasonally Adjusted' (DDDM01USA156NWDB), accessed from https://fred.stlouisfed.org. To parallel the estimation presented in Table 3, when computing the growth rate in the stock market capitalization I also bin the data by year and use the period $1990 - 2005$.

Table 4: Estimates for the knowledge spillover parameter $\bar{\lambda}$

		Avg Growth in $Q(t\|\tau)$	
ϵ	$\bar{\lambda}$	Data	Model
0.610	2.463	0.175%	0.175%
0.754	2.445	0.175%	0.175%
0.887	2.401	0.175%	0.175%

Notes: The table presents the values of $\bar{\lambda}$ that minimize the distance between the model balanced growth equilibrium and the data in regard to the average growth rate in $Q(t|\tau)$ across technologies. The first column presents different values of ϵ estimated in Table 2 for which this exercise was conducted, together with the respective calibrated value for $\bar{\lambda}$ in the second column.

Table 3, the estimated growth in patent valuation net of the aggregate stock market component equals 0.001745. On average, the growth rate in $Q(t|\tau)$ is 0.1745% across technologies.

Next, I compute the model BGP and find $\bar{\lambda}$ to minimize the distance between the model's average growth rate (across technologies) of $Q(t|\tau)$ and the empirical measure 0.1745%. Table 4 presents the results. When assuming the intermediate value $\epsilon = 0.754$ estimated in Table 2, I obtain a value for the knowledge spillover parameter equal to $\bar{\lambda} = 2.445$, with the model virtually equating the data measure for the growth rate in average quality. Table 3 also displays other values of $\bar{\lambda}$ when we use different values for ϵ estimated in Table 2. In all cases, the estimates for $\bar{\lambda}$ lie in the interval $[2.401, 2.463]$.

Finally, notice that to simulate the model and obtain $\bar{\lambda}$ to match the data, I have to take a stance on other parameters. As I now describe, the remaining parameters are externally calibrated or represent a normalization. Importantly, they have small importance to the allocation of research, and for the potential costs of misallocation, as I will highlight in the analysis that follows. Figures (C.9)-(C.11) show how the estimates for $\bar{\lambda}$ change as we change these other parameters while using our previously estimated $\epsilon = 0.754$. In all cases, the magnitudes by which $\bar{\lambda}$ changes are very small.

Other parameters

I externally calibrate the discount rate, by setting $\rho = 5\%$. Similarly, β represents the labor share in total output, as can be seen in Equation (1). I set it to $\beta = 0.66$. I normalize the labor force L, the supply of researchers R, the scale of research productivity η (which is uniform across all technologies), and the initial conditions to unit.[17] These parameters play no role in the allocation of research across technologies. The quantitative results presented next are robust to variations in each of them, with the exception of the discount rate (when I compute welfare along the transition dynamics of the model). I will, thus, report results for different calibrations of ρ.

[17] The exception is μ_0 which I normalize to 0.1.

Table 5: Baseline Calibration

Parameter	Description	Value
γ	Frontier growth rate	2%
$\bar{\lambda}$	Knowledge spillover	2.445
ϵ	Research elasticity	0.754
β	Labor share	0.66
ρ	Discount rate	5%
η	Research productivity	1
Q_0	Initial average quality	1
μ_0	Initial measure of varieties	0.1

Notes: List of model parameters and calibrated values.

Table 5 takes stock by summarizing the calibrated values for all the parameters. This is the baseline calibration I will use in Section 6 to quantify the productivity costs of research misallocation. Before that, however, I end this section by evaluating the model fit.

Model Fit Figure 10 displays both the empirical distribution of patents in the 2000 decade and the stationary age profile of innovation generated by the calibrated model. The model is successful in fitting the data, as demonstrated by the proximity of the curves. Notably, the peak of the distribution generated by the model, like in the empirical one, happens around technologies of approximately 100 years old, despite this moment being *not targeted*. For calibration, I leveraged: (i) variations in the market value of patents and the number of patents associated with distinct technologies, (ii) the age profile of patent valuations, as indicated by the growth rate of $\bar{v}(t|\tau)$ with the aging of technologies, and (iii) the overall growth rate. Targeting the peak of the empirical distribution was not part of this process.

6 Quantitative Analysis: the Costs of Research Misallocation

I now turn to the question: Does the underinvestment in new technologies predicted by the theory represent significant losses to welfare and output?

To quantify the costs of research misallocation, I start by comparing two balanced growth equilibrium (BGP) trajectories using the calibrated model. The first, also referred to as the baseline or the Laissez-faire one, is the decentralized BGP characterized in Proposition 2. With the calibrated parameters, we know that it matches the US data, including targeted and non-targeted moments, as discussed in Section 5. The second, also referred to as the efficient BGP, results from the socially optimal allocation of research chosen by a welfare maximizer planner, as discussed in Section 3.4. I compare the difference in productivity and consumption between the two scenarios, which represents the long-term gains from correcting the research misallocation.

Figure 11 shows how the socially optimal allocation of research differs from the Laissez-faire equilibrium. The three lines show the share of patents accrued to technologies of different ages.

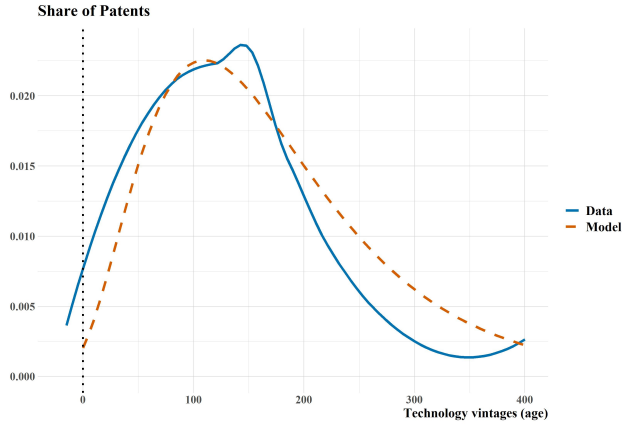

Figure 10: The Hump-Shaped Innovation-Age Profile — Model versus Data

Notes: The two lines show the share of patents accrued by technologies of different ages. In the solid (blue) line, the shares represent US data in the 2000 decade (see Section 4 for details on the data and measurement). In the dashed (orange), the shares represent the stationary distribution of research generated by the calibrated model.

The solid blue line represents the data for the US in the 2000s. Matching it, the dashed orange line shows the distribution of patents in the baseline calibration as discussed in Sections 4-5. Instead, the dot-dashed line represents the results for the optimal allocation of research.[18]

As predicted by the theory, contrary to the hump-shaped distribution observed in laissez-faire conditions (and in the data), the socially optimal number of discoveries (patents) monotonically declines as technologies age. Moreover, the share of patents accruing to the frontier technology (age zero) is drastically larger in the planner allocation than in the competitive equilibrium. As discussed in Section 3.4, this is due to the standing on the shoulders of giants' externality. Although the social value of a patent is larger than the private value within every technology, this discrepancy is larger for new technologies. As they are at the beginning of their life cycle, many follow-up inventions will yet build on the knowledge created by a patent in the present. For the older technologies, although it is true that future innovators will build on the knowledge created by current innovators, there will be only a few such future innovators. This discrepancy on spillovers implies that, even without increasing the total number of researchers, the planner can more efficiently exploit the spillovers by shifting scientists from old vintages (where spillovers are small) towards younger vintages, where spillovers are high.

Although a more efficient innovation process cannot permanently change the long-term growth rate in this economy, which is determined by the exogenous arrival of new technologies, it does determine the GDP level and the consumption frontier. Table 6 compares the level of consumption $\hat{C}$ in the different scenarios. By level of consumption, I refer to the actual value for consumption detrended by the long-run growth rate γ. In BGP trajectories, we know these levels will be constant.

[18]The patents within a given technology in period t are represented in the model by $\dot{\mu}(t|\tau) = \eta R(t|\tau)^{1-\epsilon}$, and the lines in Figure 11 represents the shares computed from this patent flow.

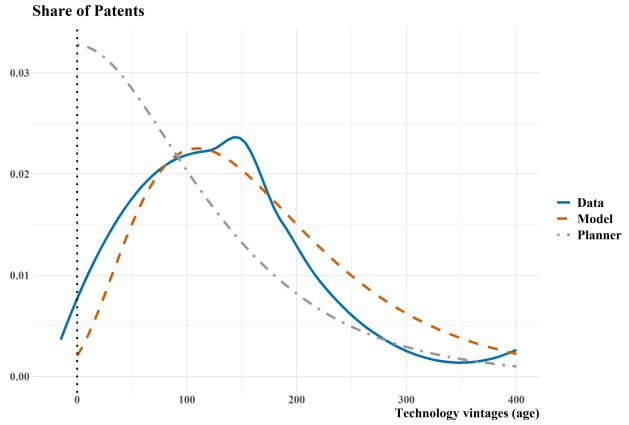

Figure 11: Optimal Research Allocation vs Market equilibrium

Notes: The three lines show the share of patents accrued by technologies of different ages. In the solid (blue) line, the shares represent US data in the 2000 decade (see Section 4 for details on the data and measurement). In the dashed (orange) and the dot-dashed (grey) lines, the shares are generated by the calibrated model (see Section 5). The former represents the decentralized BGP, and the latter the socially optimal allocation.

Table 6: Comparing the Consumption Level in Different Balanced Growth Trajectories

	(1)
$\frac{\hat{C}_{\text{efficient}}}{\hat{C}_{\text{Laissez-faire}}}$	2.0519

Notes: The table presents the ratio of consumption comparing two BGP trajectories: when the allocation of research satisfies the social optimum, $\hat{C}_{\text{efficient}}$, and when it satisfies the decentralized equilibrium $\hat{C}_{\text{Laissez-faire}}$.

Table 6 shows that, in the efficient BGP, the level of consumption is more than two times higher than the level in the baseline BGP. Correcting the research misallocation has a significant impact on the long-run consumption level and, consequently, on welfare.

6.1 Transition dynamics: from the Laissez-faire BGP to the efficient BGP

Now I turn to analyze the path connecting these two long-run balanced growth equilibriums. Although the efficient allocation of research doubles consumption in the long run, a concern is that it may take very long for such gains in output to be realized. In the short run, the growth rate may decrease as we shift resources to young technologies that are not yet very productive. Therefore, it is not obvious whether the change is welfare improving.

Exploiting the tractability of the model's transition dynamics, I compute a counterfactual trajectory in which, starting from the decentralized BGP, we impose once and for all the stationary efficient allocation of research depicted in Figure 11. Given the laws of motions for the evolution of technologies described in Equations (18)-(17), we can iterate forward in time and compute the entire equilibrium trajectory for the economy state variables.

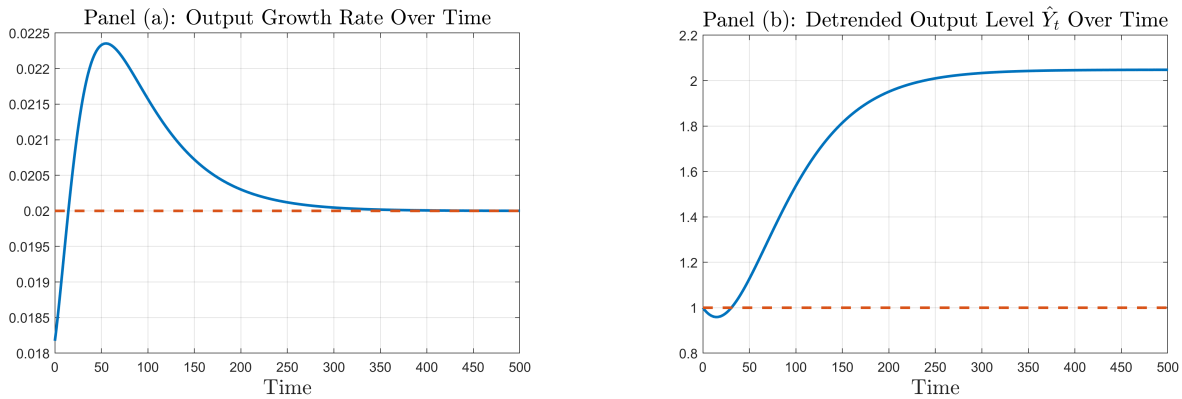

Figure 12: Transitional Dynamics - Imposing the Efficient Research Allocation

Notes: Both plots present the results of a shock that once and for all implements the stationary optimal allocation of research depicted in Figure 11, starting from the decentralized BGP. Panel (a) shows the trajectory of the output growth rate, with the dashed line representing the exogenous rate at which technologies arrive, $\gamma = 2\%$, which determines the long-run growth rate. Panel (b) plots the detrended level of output $\hat{Y}_t = Y_t \exp\{-\gamma t\}$ normalized by the initial steady state level $\hat{Y}_{\text{Laissez-faire}}$.

In the left panel of Figure 12, the solid blue line traces the trajectory of the growth rate of output resulting from this counterfactual exercise. The dashed line represents the exogenous long-run growth rate of the economy, $\gamma = 2\%$, given by the arrival of better technologies in the frontier. When the optimal stationary distribution of research is imposed, starting from the decentralized equilibrium BGP, we observe at first a temporary slowdown: the growth rate immediately jumps from 2% to approximately 1.8% and remains below 2% for 15 years. Intuitively, instead of creating varieties linked to technologies that are already very productive (and, as a result, detain large market share in the economy), researchers start to create more varieties linked to new technologies, which have not been perfected, have lower productivity, and market share. This impacts growth negatively at first. However, after 15 years, growth under the counterfactual scenario becomes higher than under Laissez-faire. In the following century, the average annual growth rate is approximately 2.2%, which compares to only 2% under Laissez-faire. This compounds in the long run to double output (and consumption) levels, as shown in Panel (b) of Figure 12.

Importantly, correcting the misallocation of research improves welfare, even when we take into account the long transition dynamics and the initial slowdown in growth. Table 7 presents the welfare gains in consumption equivalent units associated with this counterfactual scenario. How sizeable the gains are hinges on the discount rate and, hence, on the household's willingness to accept an initial growth slowdown in exchange for a permanent doubling of its consumption level. Under a standard discounting rate, $\rho = 0.025$, the consumption equivalent welfare gain equals 10%. This means that the representative household is indifferent between the trajectory with the stationary optimal allocation of research and an alternative trajectory in which consumption would remain growing at 2% for all periods but at a level 10% higher compared to the Laissez-faire trajectory. With a lower discount, $\rho = 0.01$, the welfare gains can reach 34%, and with a more

Table 7: The Welfare Effects from Research Misallocation

Discount rate	$\rho = 0.01$	$\rho = 0.025$	$\rho = 0.05$
Welfare Gain	34%	10%	0.6%

Notes: The Table shows the consumption equivalent welfare gains associated with the adoption of the stationary allocation of research. In particular, it shows how higher (%) the level of consumption in the original Laissez-faire trajectory would have to be to make the household indifferent to the trajectory in which the optimal stationary research allocation is once and for all implemented.

conservative choice, $\rho = 0.05$, the welfare gains are still positive but lower than 1%.

These results underscore the quantitative relevance of the underinvestment in emerging technologies. First, notice that the welfare results in Table 7 represent a lower bound, as the planner would not necessarily want to impose once and for all the stationary distribution of research but rather gradually adopt it in order to avoid the initial slowdown. Second, a model in which policies cannot permanently affect the growth rate of the economy makes it harder for policies to have significant welfare impacts. Imagine, for example, if the increase in the level of productivity obtained when we correct the misallocation of research increases the growth rate at which the frontier expands. This would lead to even more gains in welfare. Third, this is a model in which the total real research inputs (number of scientists) are fixed, which also naturally limits welfare changes. One possible response to a more efficient allocation of R&D would be more R&D investment, and this would make for a larger welfare effect. The fact that we obtain positive welfare results, even under a discount rate $\rho = 0.05$, despite all these conservative circumstances, shows the quantitative potential of correcting the R&D misallocation to improve welfare and productivity.

These results highlight the quantitative significance of the R&D misallocation identified in this paper. Note that the welfare results in Table 7 represent a lower bound, as a planner might prefer to gradually adopt the stationary distribution of research rather than imposing it all at once. Also, in a semi-endogenous growth model, where policies cannot permanently affect the economy's growth rate, it is challenging for policies to have substantial welfare impacts. If correcting the research misallocation were to increase the growth rate of the technological frontier, this would lead to even greater welfare gains. Furthermore, in this model, the total real research inputs (number of scientists) are fixed, which also naturally limits changes in welfare. If, in response to a more efficient R&D allocation, the total investment in innovation were to increase, this would also lead to a larger welfare effect. Despite these conservative assumptions, the fact that we observe positive welfare gains, even with a discount rate of $\rho = 0.05$, underscores the potential of correcting the misallocation identified in this paper.

7 Conclusion

It is common practice for national and international research agencies, such as the National Science Foundation or the European Research Council, to sponsor dedicated funding programs for the advancement of cutting-edge research in new technological paradigms. Despite such policies, the existing economic literature on innovation and growth has yet to reveal a reasoning behind this selective approach or the potential biases within market forces that might lead to the disproportionate allocation of research efforts towards established technologies. Knowledge externalities are a standard argument for the public funding of research, especially, basic research (Nelson, 1959; Akcigit et al., 2020). However, in standard endogenous growth models (Romer, 1990; Grossman and Helpman, 1991; Aghion and Howitt, 1992) knowledge externalities are assumed to be homogeneous, thereby muting the issue of determining the relative desirability of different types of innovation from a societal perspective. This paper addresses this gap by presenting a novel innovation-driven growth model incorporating vintage technologies, elucidating how the equilibrium distribution of R&D efforts across technologies of various ages emerges and why this allocation fails to be efficient. The analysis provides novel insights, revealing an intrinsic misallocation of R&D resources: the market allocation of research disproportionately favors aging technologies. Correcting this misallocation has the potential to stimulate productivity, even with no change in the overall research effort. The market failure arises from variations in the extent of knowledge externalities among technologies at different stages of maturity. Notably, the theory identifies a mechanism by which the Standing on the shoulders of giants' effect is more pronounced in recently emerged technologies. In the decentralized equilibrium, firms overlook this asymmetric externality and overinvest in perfecting already mature technologies rather than developing new paradigms.

I use the theory to quantify the importance of the inefficiency. To discipline the parameters of the model, I draw on a novel set of empirical findings, notably demonstrating a distinct hump-shaped distribution in patent flow across technologies of varying ages, which exhibits temporal stability. The analysis reveals that the misallocation of research efforts across technology vintages bears substantial consequences on aggregate productivity growth. Shifting from a Laissez-faire equilibrium to an efficient research allocation would increase the economy's average growth rate from 2% to 2.18% annually over the span of a century. This result relies on the slow-converging transitional dynamics identified throughout the analysis. The model has limitations that warrant further exploration. Key among them is understanding the inherent risks associated with emerging technologies, vital for policy assessment. A complementary argument to the one I propose in my theory is that investing in new technologies has an experimentation value that produces informational externalities. Also, gauging how various technologies either complement or replace labor is crucial, as this could influence the attractiveness of policies supporting them and the associated welfare implications. Finally, I sidestep creative destruction. Future research could study how the extent of creative destruction patterns differs across technologies of different maturity levels. Notwithstanding these limitations, I believe that the analysis produces valuable insights and that

the tractability of my framework can serve as a foundation for future applications and studies.

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A Proofs and additional derivations

A.1 Proof of Proposition 2

For each technology $\tau \in (-\infty, t]$, the laws of motion for quality and varieties constitute an ordinary differential equation system which, as I will show below, can be solved in closed form in the BGP. To recap, the equations are (in the main model specification I am assuming $\delta = 0$, which does not change the derivations below):

$$\mu'(t|\tau) = \frac{\eta}{1-\epsilon} R(t|\tau)^{1-\epsilon} - \delta\mu(t|\tau) \tag{A.38}$$

$$Q'(t|\tau) = \frac{Q(t|\tau)}{\mu(t|\tau)}(\lambda - 1)\frac{\eta}{1-\epsilon} R(t|\tau)^{1-\epsilon} \tag{A.39}$$

and

$$R(t|\tau) = \left(\frac{e^{-\gamma(t-\tau)}Q(t|\tau)}{\bar{Q}_t}\right)^{\frac{1}{\epsilon}} \tag{A.40}$$

where $\bar{Q}_t \equiv \left[\int_0^\infty \left(e^{-\gamma\tilde{a}}Q_t(\tilde{a})\right)^{\frac{1}{\epsilon}} d\tilde{a}\right]^\epsilon$. The BGP definition implies that $\bar{Q}_t$ is constant in such a path and can be denoted by $\bar{Q}$. Below I will also show that it exists and is unique.

Solving for the quality schedule $Q(t|\tau)$

Plugging (A.40) into (A.39) and isolating $\mu(t|\tau)$ leads to:

$$\mu(t|\tau) = \frac{Q(t|\tau)^{\frac{1}{\epsilon}}}{Q'(t|\tau)}(\bar{\lambda} - 1)\frac{\eta}{1-\epsilon} e^{-\frac{1-\epsilon}{\epsilon}\gamma(t-\tau)}\bar{Q}^{-\frac{1-\epsilon}{\epsilon}} \tag{A.41}$$

Differentiating with respect to time:

$$\begin{aligned}
\mu'(t|\tau) =& \frac{1}{\epsilon}Q(t|\tau)^{\frac{1-\epsilon}{\epsilon}}(\bar{\lambda} - 1)\frac{\eta}{1-\epsilon} e^{-\frac{1-\epsilon}{\epsilon}\gamma(t-\tau)}\bar{Q}^{-\frac{1-\epsilon}{\epsilon}} \\
&- \frac{Q(t|\tau)^{\frac{1}{\epsilon}}Q''(t|\tau)}{Q'(t|\tau)^2}(\bar{\lambda} - 1)\frac{\eta}{1-\epsilon} e^{-\frac{1-\epsilon}{\epsilon}\gamma(t-\tau)}\bar{Q}^{-\frac{1-\epsilon}{\epsilon}} \\
&- \frac{1-\epsilon}{\epsilon}\gamma\frac{Q(t|\tau)^{\frac{1}{\epsilon}}}{Q'(t|\tau)}(\bar{\lambda} - 1)\frac{\eta}{1-\epsilon} e^{-\frac{1-\epsilon}{\epsilon}\gamma(t-\tau)}\bar{Q}^{-\frac{1-\epsilon}{\epsilon}} \\
=& \frac{Q(t|\tau)^{\frac{1}{\epsilon}}}{Q'(t|\tau)}(\bar{\lambda} - 1)\frac{\eta}{1-\epsilon} e^{-\frac{1-\epsilon}{\epsilon}\gamma(t-\tau)}\bar{Q}^{-\frac{1-\epsilon}{\epsilon}}\left[\frac{1}{\epsilon}\frac{Q'(t|\tau)}{Q(t|\tau)} - \frac{Q''(t|\tau)}{Q'(t|\tau)} - \frac{1-\epsilon}{\epsilon}\gamma\right] \\
=& \mu(t|\tau)\left[\frac{1}{\epsilon}\frac{Q'(t|\tau)}{Q(t|\tau)} - \frac{Q''(t|\tau)}{Q'(t|\tau)} - \frac{1-\epsilon}{\epsilon}\gamma\right]
\end{aligned} \tag{A.42}$$

From (A.38) and (A.39), it can also be derived that:

$$\mu'(t|\tau) = \mu(t|\tau)\left[\frac{Q'(t|\tau)}{Q(t|\tau)}\frac{1}{\lambda - 1} - \delta\right] \tag{A.43}$$

Hence, comparing (A.42) and (A.43) gives:

$$\left[\frac{1-\epsilon}{\epsilon}\gamma-\delta\right]+\left[\frac{1}{\lambda-1}-\frac{1}{\epsilon}\right]\frac{Q'(t|\tau)}{Q(t|\tau)}+\frac{Q''(t|\tau)}{Q'(t|\tau)}=0$$

Integrating both sides:

$$\left[\frac{1-\epsilon}{\epsilon}\gamma-\delta\right]\int_\tau^t ds+\left[\frac{1}{\lambda-1}-\frac{1}{\epsilon}\right]\int_\tau^t\frac{Q'(s|\tau)}{Q(s|\tau)}ds+\int_\tau^t\frac{Q''(t|\tau)}{Q'(t|\tau)}=0$$
$$\left[\frac{1-\epsilon}{\epsilon}\gamma-\delta\right](t-\tau)+\left[\frac{1}{\lambda-1}-\frac{1}{\epsilon}\right]\left[\ln Q(t|\tau)-\ln Q(\tau|\tau)\right]+\ln Q'(t|\tau)-\ln Q'(\tau|\tau)=0$$

Therefore:

$$e^{\left(\frac{1-\epsilon}{\epsilon}\gamma-\delta\right)(t-\tau)}Q(t|\tau)^{\frac{1}{\lambda-1}-\frac{1}{\epsilon}}Q'(t|\tau)=c_\tau \tag{A.44}$$

where $c_\tau\equiv Q(\tau|\tau)^{\frac{1}{\lambda-1}-\frac{1}{\epsilon}}Q'(\tau|\tau)$. Fixing a given τ as we have been doing, equation (A.44) is a separable first order differentiation equation in time t, and can be rearranged as:

$$\int_{Q(\tau|\tau)}^{Q(t|\tau)}Q^{\frac{1}{\lambda-1}-\frac{1}{\epsilon}}dQ=c_\tau\int_\tau^t e^{-\left(\frac{1-\epsilon}{\epsilon}\gamma-\delta\right)(t-\tau)}dt$$

$$\frac{Q(t|\tau)^{\frac{1}{\lambda-1}-\frac{1-\epsilon}{\epsilon}}-Q(\tau|\tau)^{\frac{1}{\lambda-1}-\frac{1-\epsilon}{\epsilon}}}{\frac{1}{\lambda-1}-\frac{1-\epsilon}{\epsilon}}=c_\tau\frac{1-e^{-\left(\frac{1-\epsilon}{\epsilon}\gamma-\delta\right)(t-\tau)}}{\frac{1-\epsilon}{\epsilon}\gamma-\delta}$$

$$Q(t|\tau)^{\frac{1}{\lambda-1}-\frac{1-\epsilon}{\epsilon}}=\frac{\frac{1}{\lambda-1}-\frac{1-\epsilon}{\epsilon}}{\frac{1-\epsilon}{\epsilon}\gamma-\delta}c_\tau\left[1-e^{-\left(\frac{1-\epsilon}{\epsilon}\gamma-\delta\right)(t-\tau)}\right]+Q(\tau|\tau)^{\frac{1}{\lambda-1}-\frac{1-\epsilon}{\epsilon}}$$

Using (A.41), c_τ can be expressed as:

$$c_\tau=\frac{Q(\tau|\tau)^{\frac{1}{\lambda-1}}}{\mu(\tau|\tau)}(\bar{\lambda}-1)\frac{\eta}{1-\epsilon}\bar{Q}^{-\frac{1-\epsilon}{\epsilon}}$$

Hence:

$$Q(t|\tau)=\left[\frac{\frac{1}{\lambda-1}-\frac{1-\epsilon}{\epsilon}}{\frac{1-\epsilon}{\epsilon}\gamma-\delta}\frac{Q(\tau|\tau)^{\frac{1}{\lambda-1}}}{\mu(\tau|\tau)}\frac{(\bar{\lambda}-1)\eta}{1-\epsilon}\bar{Q}^{-\frac{1-\epsilon}{\epsilon}}\left(1-e^{-\left(\frac{1-\epsilon}{\epsilon}\gamma-\delta\right)(t-\tau)}\right)+Q(\tau|\tau)^{\frac{1}{\lambda-1}-\frac{1-\epsilon}{\epsilon}}\right]^{\frac{1}{\frac{1}{\lambda-1}-\frac{1-\epsilon}{\epsilon}}} \tag{A.45}$$

With the change of variables $a=t-\tau$ and notation $Q_t(a)=Q(t|t-a)$, we can write:

$$Q_t(a)=\left[\frac{\frac{1}{\lambda-1}-\frac{1-\epsilon}{\epsilon}}{\frac{1-\epsilon}{\epsilon}\gamma-\delta}\frac{Q_{t-a}(0)^{\frac{1}{\lambda-1}}}{\mu_{t-a}(0)}\frac{(\bar{\lambda}-1)\eta}{1-\epsilon}\bar{Q}^{-\frac{1-\epsilon}{\epsilon}}\left(1-e^{-\left(\frac{1-\epsilon}{\epsilon}\gamma-\delta\right)a}\right)+Q_{t-a}(0)^{\frac{1}{\lambda-1}-\frac{1-\epsilon}{\epsilon}}\right]^{\frac{1}{\frac{1}{\lambda-1}-\frac{1-\epsilon}{\epsilon}}} \tag{A.46}$$

Under the assumption that $Q(\tau|\tau)=Q_0$ and $\mu(\tau|\tau)=\mu_0$ for every τ (which is equivalent to

say $Q_t(0) = Q_0$ and $\mu_t(0) = \mu_0$ for every t), (A.45) gives the BGP solution for $Q(t|\tau)$ presented in (21). From (A.46) it also becomes clear how the quality schedule is stationary, depending on t and τ only through their difference: $Q_t(a)$ can be expressed, for every t, as simply $Q(a)$ without loss of generality.

A.2 Proof of Corollary 3.1

Stationary age profiles imply that output grows at a constant rate equal to γ:

$$Y_t = \frac{A_t L}{1-\beta}\int_0^\infty e^{-\gamma a} Q_t(a)\mu_t(a)da = \frac{A_t L}{1-\beta}\int_0^\infty e^{-\gamma a} Q(a)\mu(a)da \propto A_t$$

From the resource constraint, consumption is proportional to output. Hence, it also grows at the rate γ, and its detrended level equals:

$$\hat{C} = \beta L \kappa \int_0^\infty e^{-\gamma a} Q(a)\mu(a)da$$

The expression $\int_{t0}^\infty e^{-\rho t}\ln C_t dt = B_0 + B_1 \ln \hat{C}$ trivially results from writing $\ln C_t = \gamma t + \ln \hat{C}$ followed by integration by parts.

A.3 Proofs of Proposition 3 and 4

From Corollary 3.1, maximizing the household present discounted utility, as of time t_0, is equivalent to maximizing $\hat{C}$. Using the resource constraint, such a problem can be written as:

$$\underset{[R(a)]_{a>0}}{\text{Max}}\ \hat{C} = \kappa \int_0^\infty e^{-\gamma a} Q(a)\mu(a)da \tag{A.47}$$

s.t.

$$Q'(a) = Q(a)(\bar{\lambda}-1)\frac{\eta}{1-\epsilon}\frac{R(a)^{1-\epsilon}}{\mu(a)} \tag{A.48}$$

$$\mu'(a) = \frac{\eta}{1-\epsilon}R(a)^{1-\epsilon} \tag{A.49}$$

$$1 = \int_0^\infty R(a)da \tag{A.50}$$

where $Q(0) = Q_0$ and $\mu(0) = \mu_0$. The presence of the constraint (A.50) can be treated by introducing a new state variable and solving the modified maximization problem, as common in the solution to isoperimetric problems. Define $\Gamma(a) \equiv -\int_0^a R(a')da'$. Then:

$$\lim_{a\to\infty}\Gamma(a) = -1 \tag{A.51}$$

$$\Gamma'(a) = -R(a) \tag{A.52}$$

The problem at hand consists of maximizing the objective function below (which follows from

integrating by parts (A.47)):

$$\kappa \int_0^\infty e^{-\gamma a} Q(a) R(a)^{1-\epsilon} da \tag{A.53}$$

A collection of functions $[R(a), Q(a), \mu(a), \Gamma(a)]$ is admissible in this problem if the integral in (A.53) converges and the following conditions are satisfied:

$$Q'(a) = Q(a)(\bar{\lambda} - 1)\frac{\eta}{1-\epsilon}\frac{R(a)^{1-\epsilon}}{\mu(a)} \tag{A.54}$$

$$\mu'(a) = \frac{\eta}{1-\epsilon} R(a)^{1-\epsilon} \tag{A.55}$$

$$\Gamma'(a) = -R(a) \tag{A.56}$$

$$Q(0) = Q_0;\ \mu(0) = \mu_0;\ \Gamma(0) = 0 \qquad \text{(Initial conditions)}$$

$$\lim_{a\to\infty} \Gamma(a) = -1;\ \lim_{a\to\infty} Q(a) \geq Q_0 \text{ and } \lim_{a\to\infty} \mu(a) \geq \mu_0 \qquad \text{(Terminal conditions)}$$

The terminal conditions above use the fact that the limits for $Q(a)$ and $\mu(a)$ exist and are finite within the set of admissible functions. This is not ex-ante trivial (notice how it rules out divergence to ∞ or oscillating trajectories) and is proved in the following Lemma.

Lemma 1. *(Convergence of μ and Q)* $\lim_{a\to\infty} \mu(a)$ *and* $\lim_{a\to\infty} Q(a)$ *exist and are finite.*

*Proof.*From (A.55):

$$\lim_{a\to\infty} \mu(a) = \mu_0 e^{\int_0^\infty g_\mu(s) ds} \tag{A.57}$$

where $g_\mu(s) = \frac{\eta}{1-\epsilon}\frac{R(s)^{1-\epsilon}}{\mu(s)}$. Let $f(s) = \frac{\eta}{1-\epsilon}\frac{R(s)}{\mu_0}$ and notice that:

$$\int_0^\infty f(s) ds = \frac{\eta}{1-\epsilon}\frac{1}{\mu_0}$$

Since $\int f(s)ds$ converges and $0 \leq g_\mu(s) \leq f(s)$ for every s, it follows that $\int_0^\infty g_\mu(s)ds$ also converges, and hence, from (A.57), one can conclude that $\lim_{a\to\infty} \mu(a)$ exists and is finite.

Finally, notice that $g_Q(a) = g_\mu(a)(\bar{\lambda} - 1)$. Then, $\int_0^\infty g_Q(s)ds$ converges, $\lim_{a\to\infty} Q(a)$ exists and is finite. □

The problem of finding an admissible collection of functions $[R(a), Q(a), \mu(a), \Gamma(a)]$ that maximizes (A.53) lends itself to the Pontryagin's maximum principle (see Seierstad and Sydsæter (1987)). Define the Hamiltonian of the problem as:

$$H = e^{-\gamma a} Q(a) R(a)^{1-\epsilon} \kappa + \pi_Q(a) Q(a)(\bar{\lambda} - 1)\eta \frac{R(a)^{1-\epsilon}}{\mu(a)} + \pi_\mu(a)\eta R(a)^{1-\epsilon} - \pi_\Gamma(a) R(a) \tag{A.58}$$

where $\pi_Q(a)$ and $\pi_\mu(a)$ are the coestate functions, representing the shadow values of relaxing constraints (A.54) and (A.55), respectively.

An admissible solution for $[R(a), Q(a), \mu(a), \Gamma(a)]$ satisfies the necessary conditions:

$$R(a)^{\epsilon}\pi_{\Gamma} = (1-\epsilon)\left(e^{-\gamma a}Q(a)\kappa + \pi_Q(a)\frac{Q(a)}{\mu(a)}\eta(\bar{\lambda}-1) + \pi_{\mu}(a)\eta\right) \tag{A.59}$$

$$\frac{d\pi_Q(a)}{da} = -\frac{\eta\beta L\bar{\lambda}}{\delta+\gamma}\frac{\kappa}{1-\epsilon}e^{-\gamma a}R(a)^{1-\epsilon} - \pi_Q(a)(\bar{\lambda}-1)\frac{\eta}{1-\epsilon}\frac{R(a)^{1-\epsilon}}{\mu(a)} \tag{A.60}$$

$$\frac{d\pi_{\mu}(a)}{da} = \pi_Q(a)\frac{Q(a)}{\mu(a)}(\bar{\lambda}-1)\frac{\eta}{1-\epsilon}\frac{R(a)^{1-\epsilon}}{\mu(a)} \tag{A.61}$$

Notice that $\pi_{\Gamma}(a) = \pi_{\Gamma}$ is a constant since $\partial_{\Gamma}H = 0$. The terminal condition for $\Gamma(a)$, corresponding to the scientist's resource constraint (A.56), allows us to substitute π_{Γ} out when necessary. Moreover, the problem meets the conditions from Michel (1982) and Seierstad and Sydsæter (1987) for an additional necessary (transversality) condition to be established:

$$\lim_{a\to\infty} H(a) = 0 \tag{A.62}$$

This leads to the following Lemma - which, beyond providing additional information on the optimal solution, will be useful later.

Lemma 2. *(Innovation in old technologies dies in the long-run) In a solution, it must be that:*

$$\lim_{a\to\infty} R(a) = 0$$

*Proof.*The result follows from combining the transversality condition (A.62) and (A.59):

$$\lim_{a\to\infty} H(a) = 0$$
$$\lim_{a\to\infty} R(a)^{1-\epsilon}\left(e^{-\gamma a}Q(a)\kappa + \pi_Q(a)\eta\frac{Q(a)}{\mu(a)}(\bar{\lambda}-1) + \eta\pi_{\mu}(a)\right) - \phi_{\Gamma}R(a) = 0$$
$$\lim_{a\to\infty} \frac{\epsilon}{1-\epsilon}R(a) = 0$$

□

At this point, I can prove:

Proof.(*Decentralized equilibrium is efficient if* $\bar{\lambda} = 1$) If $\bar{\lambda} = 1$, the constraint (A.54) in the maximazation problem discussed above disappears, and $Q(a) = Q(0)$ for every a. Without the term $\pi_Q(a)Q(a)(\bar{\lambda}-1)\eta\frac{R(a)^{1-\epsilon}}{\mu(a)}$ in the Hamiltonian (A.58), it turns out that the necessary condition (A.61) becomes now:

$$\frac{d\pi_{\mu}(a)}{da} = 0$$

which implies $\pi_{\mu}(a) = K$, where K is a constant. Then, the transversality condition (A.62), together with Lemmas 1 and 2 imposes $K = 0$.

The first order condition (A.59) can thus be rewritten as:

$$R(a)^{\epsilon} = \frac{e^{-\gamma a}Q(a)}{\bar{Q}}$$

where $\bar{Q} = \int_0^\infty e^{-\gamma a'}Q(a')da'$. For any given age profile $[Q(a), \mu(a)]$, this corresponds to the decentralized equilibrium allocation of innovation for $R(a)$. As $Q(0) = Q_0$ and $\mu(0) = \mu_0$ in the decentralized and in the optimal path, the whole age profile $[Q(a), \mu(a), R(a)]$ will coincide in both solutions. It is also possible to explicitly solve for $\bar{Q}$ in such a solution given that $Q(a) = Q_0$ for every a. In fact, $R(a) = \gamma e^{-\gamma a}$. □

I now move forward to finish proving all the statements contained in the remaining propositions.

Lemma 3. *The limit* $\lim_{a\to\infty} \pi_Q(a)$ *exists, is finite and is non-negative.*

*Proof.*The coestate variable $\pi_Q(a)$ has a continuous path and satisfies the necessary condition (A.60), a linear first-order differential equation. As such, it can be expressed as:

$$\pi_Q(a)Q(a) = \left(K - \int_0^a \kappa e^{-\gamma s}Q(s)R(s)^{1-\epsilon}ds\right) \tag{A.63}$$

where K is the constant of integration. Hence:

$$\lim_{a\to\infty} \pi_Q(a) = \frac{1}{Q^*}\left(K - \int_0^\infty \kappa e^{-\gamma s}Q(s)R(s)^{1-\epsilon}ds\right) \leq \infty \tag{A.64}$$

where $Q^* \equiv \lim_{a\to\infty} Q(a)$, which exists and is finite by Lemma 1 - and the inequality follows from the fact that the integral above converges for all admissible functions: notice how it equals the objective function (A.53) of the problem, which in turn equals the consumption level (up to a constant of difference) and converges.

To prove the last claim, let the optimized value of the problem with initial conditions $Q(a)$ and $\mu(a)$, objective function $\int_a^\infty e^{-\gamma a'}Q(a')R(a')^{1-\epsilon}da'$, and the constraints (A.54)-(A.56) be denoted by:

$$V(a, Q(a), \mu(a)) = \kappa \int_a^\infty e^{-\gamma a'}\hat{Q}(a')\hat{R}(a')^{1-\epsilon}da'$$

where $\hat{x}(\cdot)$ denotes the optimal choice for the function $x(\cdot)$ among all feasible alternatives. The exact same steps used in the proof of Theorem 7.9 in Acemoglu (2009) and omitted here allow one to write:

$$\pi_Q(a) = \frac{\partial V(Q(a), \mu(a))}{\partial Q}$$

In words, $\pi_Q(a)$ measures the value of a marginal increase in the state $Q(a)$, of a relaxation in the constraint (A.54), the shadow value of $Q(a)$ (Acemoglu, 2009). I claim that:

$$\frac{V(a, Q(a), \mu(a))}{\partial Q} \geq 0 \quad \forall a \geq 0$$

Let $[\hat{R}(a')]_{a'}$ be the optimal choice of research allocation when the initial conditions are Q and μ. For an arbitrary $\epsilon > 0$, consider the problem with initial conditions $Q + \epsilon$ and μ. As the supply of scientists does not change with a marginal increase in the initial condition, $[\hat{R}(a')]_{a'}$ is still a feasible allocation. Observe that, by choosing it, since the initial condition for μ is unchanged, (A.54)-(A.55) show that the resulting sequence $[Q(a')]'_a$ under the new scenario must satisfy $Q(a') \geq \hat{Q}(a')$ for every a. It follows that:

$$V(a, Q + \epsilon, \mu) \geq \kappa \int_a^\infty e^{-\gamma a'} \hat{Q}(a') \hat{R}(a')^{1-\epsilon} da' = V(a, Q, \mu)$$

Rearranging terms and taking $\epsilon \to 0$, gives $\partial_Q V(a, Q, \mu) \geq 0$ for every a, as claimed. In turn, this implies that $\pi_Q(a) \geq 0$ for every $a \geq 0$, and hence it must be that $\lim_{a\to\infty} \pi_Q(a) \geq 0$.

Finally, notice that □

Lemma 4. *The limit* $\lim_{a\to\infty} \pi_\mu(a)$ *exists, is finite, and is non-positive.*

*Proof.*Lemmas 2 together with the first order condition (A.59) imply that:

$$\lim_{a\to\infty} \pi_Q(a)\eta(\bar{\lambda} - 1)\frac{Q(a)}{\mu(a)} + \eta\pi_\mu(a) = 0$$

In turn, Lemmas 1 and 3 imply that:

$$\lim_{a\to\infty} \pi_Q(a)(\bar{\lambda} - 1)\frac{Q(a)}{\mu(a)} = G$$

where G is a constant satisfying $0 \leq G < \infty$.

It follows that, necessarily:

$$\lim_{a\to\infty} \pi_\mu(a) = -G \leq 0 \tag{A.65}$$

□

From (A.63) and (A.64):

$$\begin{aligned}
\lim_{a\to\infty} \pi_Q(a) &= \frac{1}{Q^*}\left(K - \int_0^\infty \kappa e^{-\gamma s} Q(s) R(s)^{1-\epsilon} ds\right) \leq \infty \\
K &= Q^* \pi_Q^* + \int_0^\infty \kappa e^{-\gamma s} Q(s) R(s)^{1-\epsilon} ds \\
\pi_Q(a) Q(a) &= \left(K - \int_0^a \kappa e^{-\gamma s} Q(s) R(s)^{1-\epsilon} ds\right) \\
\pi_Q(a) Q(a) &= e^{-\gamma a} \int_a^\infty \kappa e^{-\gamma(s-a)} Q(s) R(s)^{1-\epsilon} ds + cte
\end{aligned}$$

where $cte \geq 0$ is a non-negative constant.

Hence, using the definition $\mathcal{E}_1(a) = \pi_Q(a)\frac{Q(a)}{\mu(a)}\eta(\bar{\lambda} - 1)$, it is straightforward to verify that $\mathcal{E}_1(a) \geq 0$ and $\mathcal{E}_1'(a) \leq 0$.

Finally, notice from (A.61) and from Lemma 4 that

$$\pi_\mu(a) = -\int_a^\infty \mathcal{E}_1(s) g_\mu(s) ds - cte2$$

where $cte2 \geq 0$ is a non-negative constant. It is straightforward to see that $\mathcal{E}_2(a) \leq 0$ and $\mathcal{E}_2'(a) \geq 0$ (the last condition can also be directly verified from (A.61) with the results derived for $\mathcal{E}_1(a)$.

One last final point concerning Proposition 4 is now proved: namely the fact that the optimal allocation of research features $R'(a) < 0$. Differentiating (A.59) with respect to a and using (A.60)-(A.61), the following expression can be derived:

$$R(a)^\epsilon \propto -\frac{e^{-\gamma a} Q(a)}{\epsilon g_R(a)}$$

where $g_R(a) \equiv \dot{R}(a)/R(a)$. Hence, as $0 \leq R(a) < \infty$ for every $a > 0$, we must have $g_R(a) < 0$.

B Foundations of the model, derivations of the laws motion and of output

Here, I first formally define the measure of varieties μ and the average quality Q associated with each technology vintage. Then, in Section B.1, I derive their laws of motion. Finally, in B.2 I derive the equilibrium expression of output (5), moving from the variety space to the technology space.

The economy modeled in Section 3 contains an evolving set Ω_t of intermediate varieties, where each variety $\omega \in \Omega_t$ is linked to one and only one technology vintage $\tau \in \mathcal{T}$ and has its its own quality level $q \in \mathcal{Q}$. One can write $\omega(\tau, q)$ to indicate that a variety ω is linked to technology τ and has quality q. For example, its productivity can be denoted by $z(\omega(\tau, q)) = A_\tau \times q$.

At a given point in time t, there will be a measure of available intermediate varieties defined on the space $\mathcal{T} \times \mathcal{Q}$. The density of this measure is denoted by $f_t(\tau, q)$ and is not normalized to be a probability density[19]. Now consider one specific technology τ and let $\mu(t|\tau)$, as in Section 3, denote the measure of all varieties embedding it. Formally:

$$\mu(t|\tau) = \int_{\mathcal{Q}} f_t(\tau, q) dq. \tag{B.66}$$

Notice, however, that knowing how widespread a technology is in terms of the number of varieties associated with it does not tell us how good or bad, in terms of quality q, these inputs are. For this purpose, as in Section 3, denote by $Q(t|\tau)$ the average quality, at time t, of varieties linked to a technology τ. Formally:

$$Q(t|\tau) = \frac{\int_{\mathcal{Q}} q f_t(\tau, q) dq}{\mu(t|\tau)}. \tag{B.67}$$

For completeness, it is important to be precise on the space $\mathcal{T} \times \mathcal{Q}$, where varieties are dis-

[19] It does not integrate to 1, but rather to the total mass of intermediates.

tributed, representing the domain of f_t for every t. By assumption , $\mathcal{Q}$ is the set of non-negative real numbers $\mathbb{R}_+$. Moreover, recall that, every period, one new technology emerges. At the calendar time t, vintages $\tau \in (-\infty, t]$ are known. Therefore, the technology set $\mathcal{T}$ is defined as the real line $\mathbb{R}$ - with the support of the density f_t satisfying $\text{supp}(f_t) \subseteq (-\infty, t] \times \mathcal{Q}$.

B.1 Derivations of the laws motion for $\mu(t|\tau)$, $Q(t|\tau)$, $\mu_t(a)$, $Q_t(a)$

Fix a technology τ and define $F_t^\tau(q) \equiv F(\tau, q)$, where $F(\tau, q)$ is the cdf of the density f (recall that it is not a probability density, but rather integrates to the total mass of varieties). Using the innvotion possibilities frontier of the economy described in Section 3, it is possible to write:

$$F_{t+\Delta}^\tau(q) = F_t^\tau(q) + \Delta\eta R(t|\tau)^{1-\epsilon} H\left(\frac{q}{Q(t|\tau)}\right),$$

where $H(\cdot)$ is the distribution from which λ is drawn at the time of an innovation. From (B.66), one can write:

$$\begin{aligned}
\mu(t+\Delta|\tau) &= \int dF_{t+\Delta}^\tau(q) \\
&= \int dF_t^\tau(q) + \Delta\eta R(t|\tau)^{1-\epsilon} \int dH\left(\frac{q}{Q(t|\tau)}\right) \\
&= \mu(t|\tau) + \Delta\eta R(t|\tau)^{1-\epsilon},
\end{aligned}$$

where the last line used the fact that $H(\cdot)$ is a probability density and integrates to 1. Subtracting $\mu(t|\tau)$ from both sides, dividing by Δ and taking the limit $\Delta \to 0$, gives:

$$\mu'(t|\tau) = \eta R(t|\tau)^{1-\epsilon}. \tag{B.68}$$

From (B.67), one can write:

$$\begin{aligned}
Q(t+\Delta|\tau) &= \frac{\int q dF_{t+\Delta}^\tau(q)}{\mu(t+\Delta|\tau)} \\
&= \frac{\int q dF_t^\tau(q) + \Delta\eta R(t|\tau)^{1-\epsilon} \int q dH\left(\frac{q}{Q(t|\tau)}\right)}{\mu(t+\Delta|\tau)} \\
&= Q(t|\tau)\frac{\mu(t|\tau)}{\mu(t+\Delta|\tau)} + \frac{\Delta\eta R(t|\tau)^{1-\epsilon} Q(t|\tau)\bar{\lambda}}{\mu(t+\Delta|\tau)}
\end{aligned}$$

Subtracting $Q(t|\tau)$ from both sides, taking the limit $\Delta \to 0$, and using (B.68), gives:

$$Q'(t|\tau) = \frac{Q(t|\tau)}{\mu(t|\tau)}(\bar{\lambda} - 1)\eta R(t|\tau)^{1-\epsilon} \tag{B.69}$$

It remains to derive the laws of motion for $Q_t(a)$ and $\mu_t(a)$, which take into account that time

affects also the age of technologies. For brevity, as the cases of $Q_t(a)$ and $\mu_t(a)$ involve exactly the same steps, I will show below only the former. Recall that, by definition, $Q_t(a) = Q(t|t-a)$. Therefore:

$$\frac{\partial Q_t(a)}{\partial t} = Q_1(t|t-a) + Q_2(t|t-a)$$
$$\frac{\partial Q_t(a)}{\partial a} = -Q_2(t|t-a.$$

Hence:

$$\begin{aligned}\frac{\partial Q_t(a)}{\partial t} &= -\frac{\partial Q_t(a)}{\partial a} + Q_1(t|t-a) \\ &= -\frac{\partial Q_t(a)}{\partial a} + \frac{Q_t(a)}{\mu_t(a)}(\bar{\lambda}-1)\eta R_t(a)^{1-\epsilon}.\end{aligned}$$

B.2 Aggregate equilibrium GDP as a function of technologies

Recall the definition of the final good in this economy:

$$Y_t = \frac{L^\beta}{1-\beta}\int_{\Omega_t} z(\omega)x_t(\omega)^{1-\beta}d\omega.$$

We know that optimal conditions from firms and the household imply that $x(\omega) = L$. Therefore:

$$Y_t = \frac{L}{1-\beta}\int_{\Omega_t} z(\omega)d\omega.$$

Since all varieties are symetric except for their technology and quality level, we can index them by $\omega(\tau, q)$ and write:

$$\begin{aligned}Y_t &= \frac{L}{1-\beta}\int_{-\infty}^{t}\int_0^\infty z(\tau,q)f(\tau,q)dqd\tau \\ &= \frac{L}{1-\beta}\int_{-\infty}^{t}\int_0^\infty A_\tau q f(\tau,q)dqd\tau \\ &= \frac{L}{1-\beta}\int_{-\infty}^{t} e^{\gamma\tau}Q(t|\tau)\mu(t|\tau)d\tau\end{aligned}$$

where the definition of $Q(t|\tau)$ in (B.67) makes the transition from the second to the third line.

B.3 D_t as the stock market to GDP ration

Remember the value function for a variety belonging to technology τ:

$$v_t(\omega) = \int_t^\infty e^{\left(-\int_t^s r(v)\,dv\right)}\,\pi_t(\omega)\,ds = \beta z(\omega)L \times D_t, \tag{B.70}$$

where $D_t \equiv \int_t^\infty \exp\left(-\int_t^s r(v)dv\right) ds$. Then, the total value of corporate assets (value of the stock market):

$$V_t = \int_{\Omega_t} v_t(\omega)d\omega = D_t\beta(1-\beta)Y_t$$

Therefore, the stock market to GDP ratio, V_t/Y_t, is proportional to D_t, and their growth rates are the same.

B.4 Logistic trend estimation

Model the growth of the patent share of a technology with the logistic function:

$$share(t) = \frac{K}{1+e^{-(\alpha+\beta t)}}$$

where K is the ceiling value (maximum share achieved by a technology)

Transform it:

$$\log\left(\frac{share(t)}{K - share(t)}\right) = \alpha + \beta t$$

Recover $\hat{\alpha}$ and $\hat{\beta}$ from OLS.

B.5 Exogenous death shock faced by varieties

In this section, I present an extension of the theory in which intermediate varieties may be hit by a death shock - represented by a Poisson process with intensity δ. The baseline model in Section 3 corresponds to the particular case of $\delta = 0$. In the general case of $\delta \geq 0$, the value $v_t(\omega)$ of a variety $\omega \in \Omega_t$ satisfies:

$$\begin{aligned} v_t(\omega) &= \int_t^\infty e^{\left(-\int_t^s r_v + \delta\, dv\right)}\, \pi_t(\omega)\, ds \\ &= \beta z(\omega) L \times D_t \end{aligned}$$

where $D_t \equiv \int_t^\infty \exp\left(-\int_t^s r_v + \delta\, dv\right) ds$. If the interest rate r is constant, we have $D_t = (r+\delta)^{-1}$. The expected value of innovation within a given technology, $\bar{v}$, is the same as in (9), with D_t now including the death shock rate. Since it represents an equal risk to all varieties, regardless of the technology to which they are linked, it will not have an influence on the allocation of researchers, as is the case of the interest rate. Equation (13), therefore, is still valid to represent the share of researchers $R_t(a)$ working on a given technology given the current state of the economy.

The key distinction introduced by the death rate is that, despite ideas getting harder to find, research productivity is bounded above zero. In the long-run, a constant number of researchers can sustain a constant growth rate within a given technology.

The laws of motion for the evolution of varieties and average quality are now given by:

$$\partial_t \mu_t(a) = -\partial_a \mu_t(a) + \eta R_t(a)^{1-\epsilon} - \delta \mu_t(a) \tag{B.71}$$

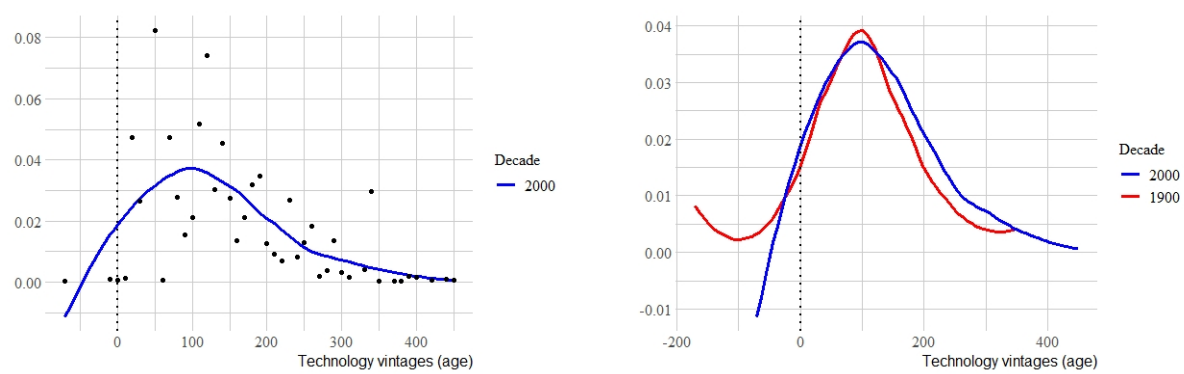
$$\partial_t Q_t(a) = -\partial_a Q_t(a) + Q_t(a)(\bar{\lambda} - 1)\eta \frac{R_t(a)^{1-\epsilon}}{\mu_t(a)}. \tag{B.72}$$

The equilibrium is characterized as in Definition 1, with the laws of motion for the state variables being now (B.71)-(B.72).

The key distinction introduced by the death rate is that research productivity is bounded above zero within a technology despite ideas getting harder to find. In the long run, a constant number of researchers can sustain a constant growth rate within it. At a given point in time, the death process removes varieties whose average quality is $Q(t|\tau)$, while innovation adds new varieties whose quality is on average $Q(t|\tau)\bar{\lambda}$. This allows the average quality to steadily grow if the number of researchers targeting τ (and hence the innovation flow) is kept constant.

C Additional figures

Figure C.1: Patenting shares per technology vintages



Notes: In computing the patenting shares corresponding to each technology, each patent is weighted by its importance indicators from Kelly et al. (2021). The age of technologies is constructed by the procedure described in section 4. The dots in the left-hand side figure represent the share of the corresponding vintages among all US patents. The solid lines represent smoothed curves to make the visualization easier.

Figure C.2: Patenting shares per technology vintages

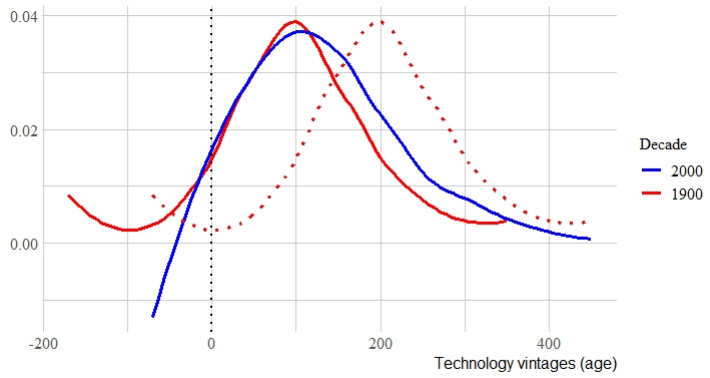


Figure C.3: Patenting shares per technology vintages

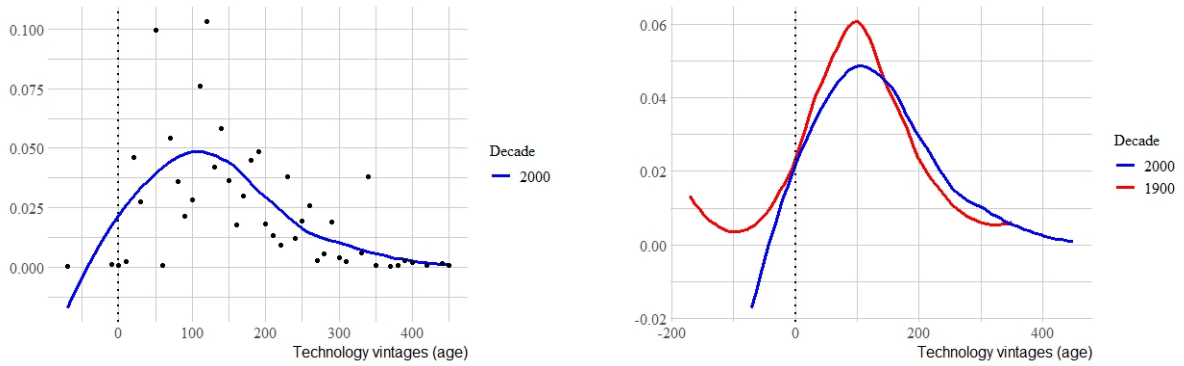


Figure C.4: Patenting shares per technology vintages

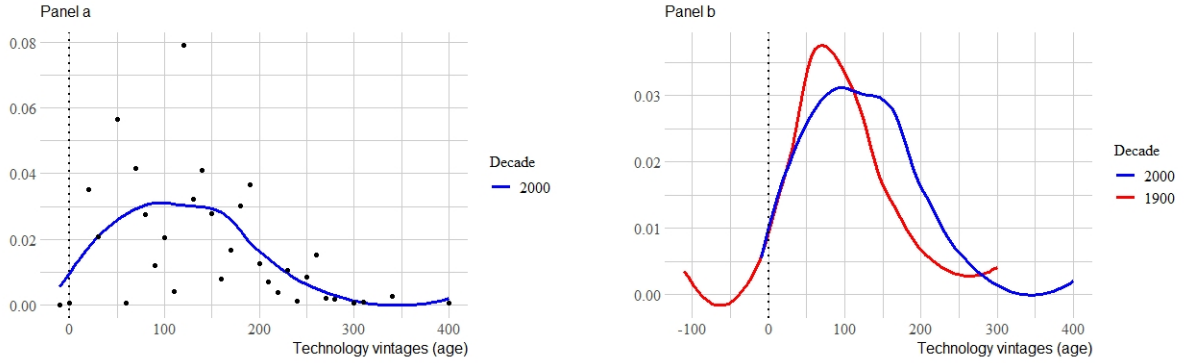


Figure C.5: Patenting shares per technology vintages

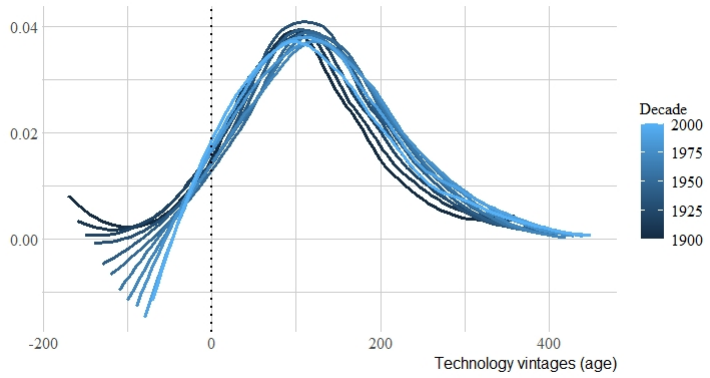


Figure C.6: Patenting shares per technology vintages

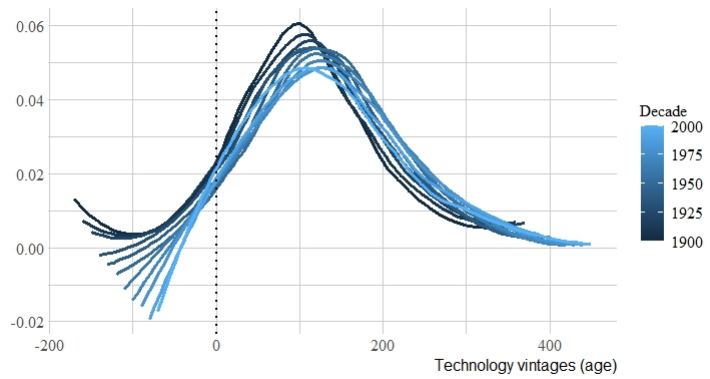


Figure C.7: Patenting shares per technology vintages

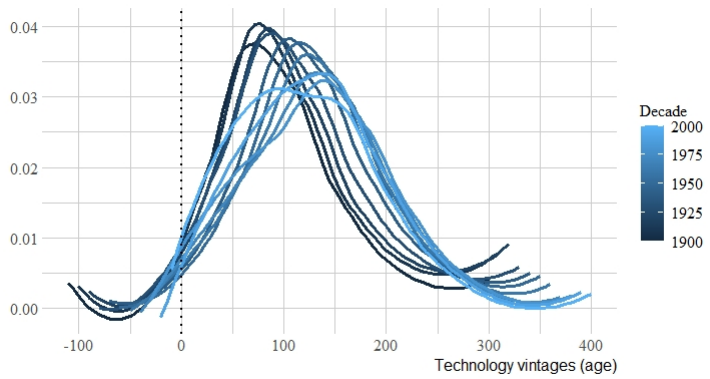


Figure C.8: Patenting share and proxied emergence date.

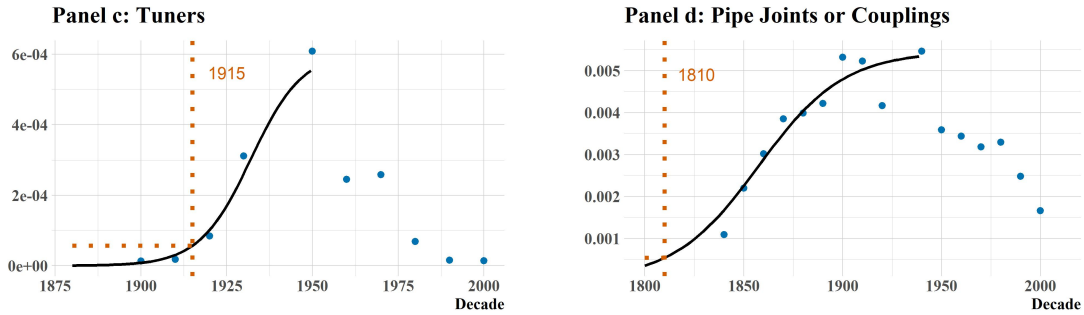


Figure C.9: Calibration sensitivity

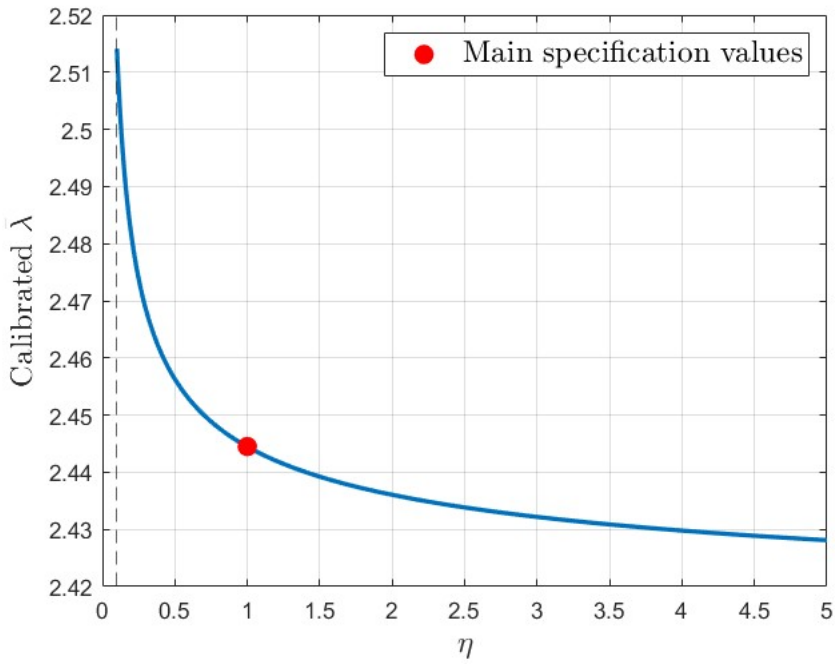


Figure C.10: Calibration sensitivity

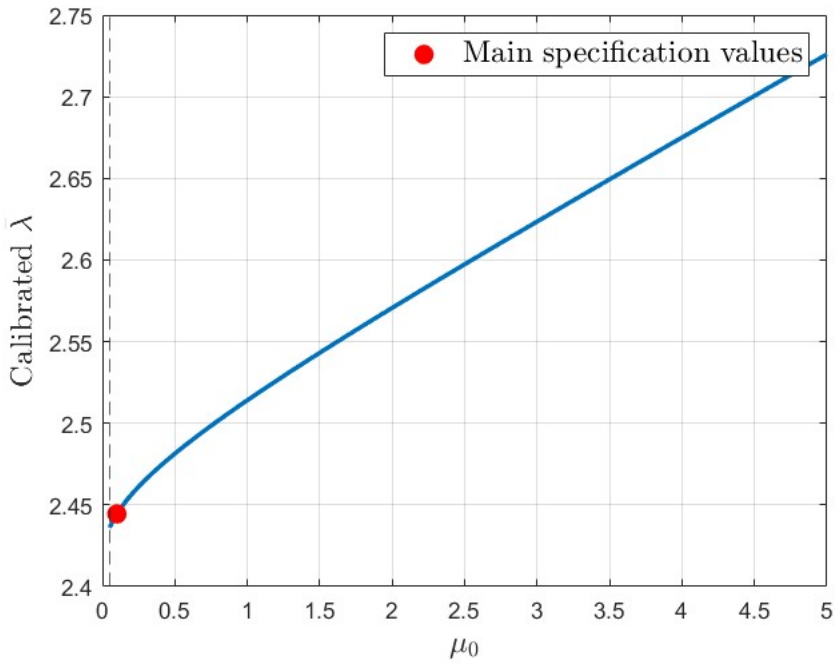


Figure C.11: Calibration sensitivity